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TREATISE

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PLANE AND SPHERICAL

TRIGONOMETRY,

IN THEORY AND PRACTICE.

ADAPTED TO THE USE OF STUDENTS.

Extracted mostly from similar works of Ludlam, Playfair,
Vince, and Bonnycastle.

BY F. NICHOLS,

Member of the Literary and Philosophical Society of Manchester,
and of the American Philosophical Society.

PHILADELPHIA:

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Fry and Kammerer, Printers.

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Nichols

Young's Analysis of Natural Philosophy, 8vo.
 Cavallo's Natural Philosophy, 4 vols. 8vo.

The following books may be read by persons who have little mathematical knowledge.

Compend of Astronomy, 12mo, published by Nichols.
 Bonnycastle's Astronomy, 8vo.
 Bryan's do. 8vo.
 Ferguson's do. 8vo.
 Biot's Elements of Physical Astronomy, 2 vols. 8vo, in French.
 La Place's System of the World, by Pond, 2 vols. 8vo.
 Young's Lectures on Natural Philosophy, vol. i. 4to.
 Gregory's Economy of Nature, 3 vols. 8vo.
 Elements of Art and Science, by Webster, 2 vols. 8vo.
 Haüy's Physics, by Gregory, 2 vols. 8vo.
 Ferguson's Lectures on Select Subjects, by Brewster, 2 vols. 8vo.
 Adams's Lectures on Natural and Experimental Philosophy, by Patterson, 4 vols. 8vo.
 Gregory's Dictionary of Arts and Sciences, 2 vols. 4to.
 Nicholson's Encyclopedia, 6 vols. 8vo.

Note. Many of the books toward the end of the list treat of the same subjects, and the last two comprise short treatises on the various branches of mathematics, natural and experimental philosophy, the useful and ornamental arts, &c.

PREFACE.

TRIGONOMETRY is a branch of the general science of geometry. It treats of the properties and relations of certain straight lines described in and about a circle, and also teaches to compute the dimensions of the sides and angles of any triangle. It supplies fundamental rules for ascertaining the distances and altitudes of objects both terrestrial and celestial. Without the aid of this science, the form and magnitude of the earth, and the magnitudes, distances, motions, and eclipses of the heavenly bodies would be utterly unknown. In geography, navigation, astronomy, and the military art, its assistance is absolutely necessary. By its unerring rules we are able to determine the latitudes and longitudes of places upon the surface of the earth, and the varying positions and appearances of the heavenly bodies; to survey countries, and to travel by sea or land to any part of the world, that is accessible to the adventurous spirit of curiosity or avarice.

One main object of the publisher is to furnish a treatise of trigonometry, plane and spherical, of a middle size between the concise elementary tracts containing the definitions and a few theorems barely necessary to the solution of the common cases of triangles, and the larger and more comprehensive works, designed for the use of mathematicians, or of students who have time, ability, and inclination to enter deeply into such curious and difficult speculations.

This treatise is the production of different authors of approved merit, whose shares may be nearly assigned as follows.

Some articles have been supplied by Legendre, Horsley, and Lacroix, which, as they are not numerous, need not be specified.

Plane trigonometry consists mostly of the lectures of an eminent mathematician, W. Ludlam, A. M., formerly a fellow and tutor of St. John's college, Cambridge. These, with

other mathematical lectures, were read by Mr. Ludlam, during many successive years, to the junior students of St. John's; and were received with a degree of approbation adequate to their merit. After the decease of that popular instructor, his lectures were published. They have passed through several editions, and are still used in the university of Cambridge, and also in many mathematical schools in Britain.

Mr. Ludlam's Rudiments of Mathematics have been long considered one of the best introductions to Algebra, to Euclid's Elements, and to Plane Trigonometry, in the English language. "In simplicity and perspicuity of style, and in clearness and accuracy of demonstration, (says a teacher of experience), this author is not inferior to his great master, the famous professor Saunderson; and in these respects is not excelled by any writer of the present time. Hence his trigonometry is admirably adapted to the use of students in our colleges and higher schools."

In his advertisement to the Elements of Plane Trigonometry, Mr. Ludlam informs his readers that "he has selected such propositions as may probably be wanted in the usual course of academical studies, especially in astronomy and natural philosophy; and that he has taken the liberty of borrowing demonstrations, and even of altering them, when he thought they could be improved."

A number of detached articles, and the greater part of the third section, are extracted from Playfair's Trigonometry. The first general division of the third section is taken from Ludlam, Vince, and Bonnycastle;* but mostly from the first.

The first and second sections contain the elements of plane trigonometry, the practical and numerical solutions of all the cases of rectilinear triangles, and the mensuration of the heights and distances of objects. These two sections are complete of themselves.

* This useful and meritorious writer appears to have followed Legendre's plan, and method of demonstration, and often translates from his Trigonometry without acknowledging the obligation.

Legendre's Trigonometry is probably equal, if not superior, to any elementary treatise of the same extent. It is uncommonly perspicuous and elegant, and is well adapted to the use of students who are previously acquainted with algebra and geometry. His Elements of Geometry and Trigonometry have not been translated from the original French; but a translation of them would make a valuable addition to our stock of recent mathematical science, which is certainly very defective.

Ludlam, who has been our principal guide through plane trigonometry, has not written on spherical; and therefore we have been obliged to follow other authors of experience and ability in the remaining part of this work. Those who have afforded us most assistance are Playfair, Vince, and Bonycastle, from whose writings the greater part of spherical trigonometry has been extracted. The works of Legendre, Horsley, Lacroix, and others, have furnished some articles of importance.

It is no small recommendation of the different works which have supplied the materials of this treatise, that the authors of them have been, for many years, public teachers of mathematics in foreign universities. He who writes for the instruction of youth ought to possess the experience and skill, which the art and practice of teaching can alone communicate. Men who have not received the benefit of a liberal education, and are not conversant in the business of tuition, are apt to form plausible theories of the discipline and economy of schools and colleges; but they are generally founded on ideal principles, and therefore cannot be reduced to rational, and successful practice.

In the third section of plane trigonometry the changes of the algebraic signs of the lines described in and about a circle, commonly denominated *trigonometrical lines*, and the principal geometrical properties of those lines, are fully explained and demonstrated. A perfect knowledge of the changes of the signs of those lines, and of their chief properties, is indispensable in the theory of central forces and astronomy, and in many parts of mechanical philosophy, especially in the resolution of physical problems. This is one of the most useful and necessary parts of trigonometry, without the knowledge of which the student can make no proficiency in the higher branches of mathematics.

The construction of trigonometrical tables is annexed to plane trigonometry, because it is a direct application of the trigonometrical theorems previously demonstrated. This and the theorems respecting the sines and cosines of arcs, "which (says professor Playfair) are the foundation of those applications of trigonometry lately introduced, with so much advantage, into the higher geometry," are extracted from Playfair's *Elements of Trigonometry*, with a few additions.

"Though the solution of triangles forms a distinct and material part of trigonometry, yet a treatise, which should be *now*

confined to that object, would be justly deemed defective. Many of the most useful formulæ can be easily obtained; and in the present state of science every treatise upon trigonometry should comprise some portion of trigonometrical analysis." British Review for March 1811.

The definitions and preliminary principles contained in the introduction to spherical trigonometry are intended to be annexed, with their demonstrations, to the second edition of Playfair's Geometry, under the title of Elements of Spherical Geometry. They properly belong to the elements of geometry, though they are usually prefixed to treatises of spherical trigonometry. With the same kind of impropriety might the common properties of rectilinear triangles be prefixed to plane trigonometry.

It may perhaps be said that this treatise of trigonometry might be taught after any other book of geometry, beside Playfair's, if spherics, or the common properties of the sphere, had been connected with it. But to this objection we may reply, that no other edition of Euclid's Elements of Geometry can continue much longer in use in any places of education where the rudiments of mathematics are systematically and successfully cultivated, *because Playfair's contains so many corrections, additions, and improvements, and is so far superior to all former editions of the celebrated work of the Greek geometer.* Indeed Playfair's Geometry is now generally used as a textbook in our colleges and higher schools; and the teachers uniformly acknowledge its utility and excellence. It can be read by students in much less time, and with far less difficulty and labour, than Simson's or any other edition of Euclid's Elements of Geometry.

Some teachers would perhaps be satisfied with the elementary propositions of trigonometry, and would be willing to dispense with their application to the numerical solution of the cases of triangles, and to the mensuration of heights and distances, &c. But this confined plan does not accord with the import and object of trigonometry, which *is the art of finding the dimensions of the sides and angles of triangles*; or, agreeably to professor Playfair's definition, "is the application of arithmetic to geometry, or, more precisely, of number to express the relations of the sides and angles of triangles to one another." Hence the nature and design of this work comprehend a theoretical and practical treatise of plane and spherical trigonometry, which shall be adapted to numerical computation; and, as far as this science is concerned, shall serve as an

introduction to mechanical philosophy and the higher branches of mathematics. Besides, when theory and practice are thus united, the work may probably be useful to some young men who have acquired a competent knowledge of the elements of algebra and geometry, but cannot obtain the assistance of a preceptor.

Rules, without suitable examples to illustrate them, are of little use to youth, because they do not make a lasting impression upon their memory, and consequently are soon forgotten. Hence the elementary propositions in this work are illustrated by a variety of numerical examples, in the solutions of all the cases of plane and spherical triangles, and in the mensuration of the heights and distances of terrestrial objects. By the solution of these numerical problems, the learner will acquire a facility in the use of the tables of logarithms, and in trigonometrical calculations, which will promote his progress in the study of navigation, physical astronomy, and other branches of mixed mathematics.

In some articles, algebra has been applied to facilitate and shorten the demonstrations, which would have been long and difficult by pure geometry. The sciences of algebra and geometry afford mutual assistance in mathematical investigations, and should be applied jointly when they cannot easily effect the object separately, or when one conduces essentially to the aid of the other. In favour of this practice we have the authority of the ablest mathematicians during the last fifty years; and algebra is constantly applied to geometry, by the most eminent writers of the present time, whenever it can be introduced with advantage into a mathematical process.

As this work contains many properties of plane and spherical triangles, which are useful, and even necessary, in certain parts of mathematics, but are not requisite in the solution of the common cases of triangles, it is proper to inform the reader what articles may be omitted entirely, or deferred till he has made some proficiency in other branches of mathematics. The articles which may be wholly omitted, or postponed to a future time, are specified in page iv.

It was the intention of the publisher to subjoin the stereographic projection of the sphere; but he found that this could not be done without increasing the size and price of the book too much. The mere principles of projection are short; but without their practical application they would be of little benefit to the learner. It was therefore judged most expedient to

reject this part of the original plan, and to refer the reader to the best works on the different projections of the sphere. See Horsley's *Mathematics*, vol. iii, Webber's *Mathematics*, vol. ii.

I do not refer to Emerson's *Treatise on the Projection of the Sphere*, because it has been remarked by Dr. Horsley and other mathematicians, that "Emerson's works are composed with great depth of knowledge, and upon sound principles; but without elegance, without order, and without perspicuity." But writings of this character do not afford models fit for the study and imitation of youth.

Of all the recent English writers, Dr. Horsley, bishop of Rochester, appears to have treated of the different projections of the sphere most fully and elaborately. His book is composed with the precision and prolixity of the ancient geometers, whose works he professes to admire. "We conceive (says he) that we have treated this subject of projection in a manner which goes to the bottom of it, brings many curious properties to light, not observed by former writers, and clears up many difficulties."*

The learner will not be duly qualified to study this treatise with advantage, if he do not possess a competent knowledge of vulgar and decimal fractions, of the elements of algebra and geometry, and of practical geometry. Furnished with those preparatory qualifications, he will be able to read the whole book without much difficulty.

Most of the demonstrations in this work are simple and plain. A few in spherical trigonometry are, from the nature of the propositions, more artificial and difficult; but they are so diffuse that a reader, who is acquainted with the elements of algebra and geometry, will easily understand them. In no book of equal extent will a learner meet so few impediments to retard his progress.

The publisher does not presume to claim this book as author, but only as editor and proprietor. He has prefixed his name in the title page, because, without a name, it could not be distinguished from other works of the same kind, and consequently could not be obtained by persons residing at a distance from the places where it is published and sold.

F. N.

Philadelphia, August 1811.

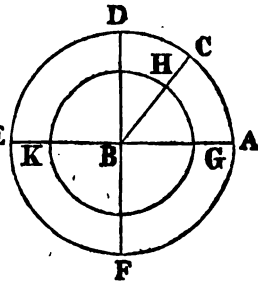
* A republication of this part of the bishop's book would probably be an acceptable present to the professors of mathematics in our colleges. It appears to be studied in the university of Oxford.

INTRODUCTION.

LEMMA I.

1. AN angle at the centre of a circle is to four right angles, as the arc on which it stands is to the whole circumference of the circle.

Let ABC be an angle at the centre of the circle ACF, standing on the arc AC; the angle ABC : four right angles :: arc AC : whole circumference ACF.



Produce AB till it meet the circumference in E, and through B draw DBF perpendicular to AE. Because ABC, ABD are two angles at the centre of the circle ACF, the angle ABC : angle ABD :: arc AC : arc AD (33. 6);* therefore the angle ABC : four times the angle ABD :: arc AC : four times the arc AD (4. 5). But ABD is a right angle, therefore four times the arc AD is equal to the whole circumference ACF. Hence the angle ABC : four right angles :: arc AC : whole circumference ACF.

2. COR. 1. Equal angles at the centres of different circles stand on arcs which have the same ratio to their circumferences.

If the angle ABC at the centre of the circles ACF, GHK, stand on the arcs AC, GH, then the arc AC : whole circumference of the circle ACF :: angle ABC : four right angles; and the arc GH : whole circumference of the circle GHK :: angle ABC : four right angles. Therefore, &c.

* The references to the elements of geometry in this work are made to Playfair's Geometry, which is a new edition of Euclid's Elements, with great additions and improvements.

3. COR. 2. Hence the arcs which subtend equal angles at the centres of different circles are to one another as the circumferences of the circles.

The arc AC : circumference ACF :: arc GH : circumference GHK (11. 5).

4. COR. 3. Hence, if the circumferences of any two circles be divided into the same number of equal parts, whatever number of those parts is contained in any arc AC of one circle, subtending a given angle ABC, the same number of parts will be contained in the arc GH of the other circle, subtending the same angle ABC.

5. COR. 4. Hence, if a circle of any radius whatever be divided into 360 equal parts, called *degrees*, each degree into 60 equal parts, called *minutes*; each minute into 60 equal parts, called *seconds*, &c., the number of degrees, minutes, and seconds, intercepted by two radii, BA, BC, will be a proper measure of the angle ABC.

For the number of degrees, minutes, &c. so intercepted, will be to 360 degrees, as the angle ABC is to four right angles; therefore the number of degrees, minutes, &c. so intercepted, will be to the number of degrees, minutes, &c. intercepted by any other two radii, as the magnitude of the former angle to the magnitude of the latter.

And this is the proper meaning of a *measure*. One quantity is said to be a *measure* of another, when the measure in one case is to the measure in any other, as the magnitude of the quantity to be measured in the former case is to the magnitude of the quantity to be measured in the latter case.

6. Since then the whole circumference of a circle consists of 360 degrees, a right angle will be 90 degrees, two right angles will be 180 degrees, half a right angle will be 45 degrees, each of the angles of an equilateral triangle will be 60 degrees.

The magnitude or quantity of any arc or angle is denoted thus, $21^{\circ} 16' 45''$, or thus, $21^{\circ} 16' 45''$, and is read 21 degrees, 16 minutes, 45 seconds.

Note. Some mathematicians advise to divide a degree into centesimal parts rather than sexagesimal; and it would perhaps be more convenient to divide not only a degree, but even the whole circumference of the circle, in a decuple ratio; which division has been adopted in France. The advantage which would accrue from the division of the circle in a decuple proportion arises chiefly from the nature of the notation now in

use, by which the value of any number is augmented in a decuple proportion, and diminished in a subdecuple proportion, to which therefore such a division would be peculiarly adapted.

The ancients chose the number 360, probably because the Egyptians, to whom this division of the circle is usually ascribed, thought that the year consisted of 360 days, or that the circle which the sun seemed to describe in a year was already distinguished by nature into 360 parts. Another convenience in this division is, that the number 360 is divisible by several numbers without a remainder, as by 10, 6, 4, 2, 90, 60, 9, &c., by which means they could accurately express several parts of the periphery of the circle in whole numbers, which they considered far more clear, and less intricate than fractions.

LEMMA II.

7. ALL angles are to one another as their arcs directly; and their radii inversely. Fig. to Art. 1.

The angle ABC : 4 right angles :: the arc AC : circumference $ADEFA$ (1.), therefore the angle $ABC = \frac{AC \times 4 \text{ right angles}}{\text{circum. } ADEFA}$. But four right angles are a constant quantity, and the circumferences of circles are as their diameters or radii (8. 1. Sup.). Consequently the angle ABC varies as $\frac{AC}{AB}$.

Let ac be any other arc, abc any other angle, ab the radius; then, in the same manner, the angle abc will vary as $\frac{ac}{ab}$.

Hence the angle ABC : angle abc :: $\frac{AC}{AB} : \frac{ac}{ab}$

TRIGONOMETRY.

8. **TRIGONOMETRY** is a branch of the general science of geometry, which treats of the properties and relations of certain straight lines drawn in and about a circle, and also teaches to compute the sides and angles of triangles by means of a set of tables called a *Trigonometrical Canon*.

It is divided into two parts, Plane Trigonometry and Spherical. The former has for its object *rectilinear* triangles, the latter, triangles formed by the intersection of three great circles on the surface of a sphere.

9. In a practical sense, Trigonometry may be defined to be, *the application of number to express the relations of the sides and angles of triangles to one another*. It therefore supposes that the learner can perform the common numerical operations; and it borrows from arithmetic certain signs or characters which peculiarly belong to that science. Thus, the product of two numbers represented by A and B, is denoted either by $A \cdot B$, or $A \times B$; and the product of two or more numbers multiplied by one or more numbers, as $A+B$ into C, or $A+B$ into $C+D$, is expressed by $(A+B) C$, and $(A+B) (C+D)$, or sometimes by $\overline{A+B} \times C$, and $\overline{A+B} \times \overline{C+D}$.

The quotient of any number A divided by any number B, is expressed by $\frac{A}{B}$, or $A \div B$.

The sign $\sqrt{}$ signifies the square root. Thus, $\sqrt{M}$ denotes the square root of M, or is a number which, if multiplied by itself, will produce the number M. Also, $\sqrt{M^2+N^2}$ denotes the square root of $M^2 + N^2$.

PLANE TRIGONOMETRY.

10. Plane trigonometry is the science which treats of the analogies of plane triangles, and of the methods of determining their sides and angles. It also comprehends whatever appertains to the properties and relations of certain straight lines drawn in and about a circle.

11. In this treatise Plane Trigonometry is divided into three sections. The first section explains the principles; the second contains the rules of calculation, the practical solution of the cases of rectilinear triangles, and the mensuration of heights and distances; the third contains the changes of the signs of trigonometrical lines, the investigation of some theorems necessary to the solution of the more difficult problems in Trigonometry, and the construction of trigonometrical tables.

SECTION 1.

ELEMENTS OF PLANE TRIGONOMETRY.

DEFINITIONS.

12. ABOUT the centre C (Fig. 1), with the radius CA , describe a circle; produce AC till it meet the circle again in D , so that AD may be a diameter. Draw the diameter HL perpendicular to AC . The two diameters AD , HL will divide the circumference into four equal arcs, called *quadrants*, each arc containing 90 degrees (6). Draw tAT touching the circle in A ; and draw the radius CB , and produce it till it meet the line tAT in T .

13. DEF. 1. The difference of any arc from a quadrant, or of any angle from 90 degrees, is called the *complement* of that arc or angle. Thus, let the arc AB be reckoned from A as its origin,* then HB is the complement of AB , and the angle HCB is the complement of ACB .

14. DEF. 2. The difference of any arc from a semicircle, or of any angle from 180 degrees, is called the *supplement* of that arc or angle. Thus, the arc DHB is the supplement of AB , and the angle DCB is the supplement of ACB .

15. DEF. 3. The *chord* of any arc is a right line drawn from one extremity of the arc to the other.

Let the points B and E be in the circle, and join B , E ; then the straight line BE is the chord of the arc BAE , or of the angle BCE , of which the arc BAE is the measure.

16. DEF. 4. The *sine* or *right sine* of any arc is a straight line drawn from the end of the arc perpendicular to the diameter passing through the beginning of the arc. From B demit BF perpendicular to AD , then BF is the sine of the arc AB , or of the angle ACB .

* A geometrical circle has no beginning. But in the application of Trigonometry to most subjects, especially to astronomy, it is necessary to fix upon some point as a beginning, whence all arcs are to be computed. Thus, the degrees on the equator and the ecliptic are reckoned from one of the equinoctial points.

Again, let the arc $ABHb$ be the supplement of AB ; draw the radius Cb , and produce it till it meet the circle in E , and the line TAt in t ; demit bf and EF perpendicular to the diameter AD ; then bf is the sine of the arc $ABHb$, or of the angle ACb , and EF is the sine of the arc $ABbDLE$, or of the angle which is measured by that arc.

17. DEF. 5. The *versed sine* of an arc is that part of the diameter passing through the beginning of the arc which is intercepted between the beginning of the arc and the right sine. Thus, AF is the versed sine of the arc AB , or of the angle ACB ; and Af is the versed sine of the arc AHb , or of the angle ACb .

18. DEF. 6. The *tangent* of an arc is a straight line touching the circle in the beginning of the arc, produced thence till it meets the radius (produced) drawn through the end of the arc. Thus, AT is the tangent of the arc AB , or of the angle ACB ; and At is the tangent of the arc AHb , or of the angle ACb .

19. DEF. 7. The *secant* of an arc is a straight line drawn from the centre through the end of the arc, and produced till it meets the tangent. Thus, CT is the secant of the arc AB , or of the angle ACB ; and Ct is the secant of the arc AHb , or of the angle ACb .

20. DEF. 8. The *cosine* of an arc is the part of the diameter passing through the beginning of the arc which is intercepted between the centre and the sine. Thus, CF is the cosine of the arc AB , or of the angle ACB ; and Cf is the cosine of the arc AHb , or of the angle ACb .

21. DEF. 9. The *cotangent* of an arc is a line touching the circle in the end of the first quadrant, produced thence till it meets the radius (produced) drawn through the end of the arc. Thus, draw HK , touching the circle in H , and meeting the radius produced in K ; then HK is the cotangent of the arc AB , or of the angle ACB .

22. DEF. 10. The *cosecant* of an arc is a straight line drawn from the centre through the end of the arc, and produced till it meets the cotangent. Thus, CK is the cosecant of the arc AB , or of the angle ACB .

Scholium to Definitions 8, 9, 10.

23. It is manifest that the cotangent and cosecant are referred to the diameter HL passing through the end of the quadrant, in like manner as the tangent and secant are referred to the diameter AD passing through the beginning of the quadrant.

It appears also, that the cosine, cotangent, and cosecant of an arc under 90 degrees, or of an angle less than a right angle, are respectively equal to the sine, tangent, and secant of the complement of that arc or angle.

Draw BI perpendicular to the diameter HL , then $CF=BI$. But BI , HK , CK , are respectively the sine, tangent, and secant of the arc HB , reckoned from H as its beginning. Now the arc HB is the complement of the arc AB , and the angle HCB is the complement of the angle ACB . Therefore the cosine, cotangent, and cosecant of any arc or angle, are respectively equal to the sine, tangent, and secant of the complement of that arc or angle, whence they derive their names.

Of the Properties and Relations of Trigonometrical Lines.

24. The sine, cosine, tangent, secant, &c. of any angle ACB , in a circle whose radius is AC (fig. 1), will be to the sine, cosine, &c. of the same angle ACB , in any other circle whose radius is AC (fig. 2), respectively as the radius of the former circle is to the radius of the latter.

For the several right-angled triangles corresponding to one another in fig. 1 and 2, having one acute angle equal in each (the angle ACB in fig. 1 equal to ACB in fig. 2), are equiangular; therefore $BF(1) : BF(2) :: BC(1) : BC(2)$. Similar analogies obtain in the case of all the other corresponding triangles, in each of which one corresponding side is radius.

25. Hence, if the radius of any circle be divided into 10,000,000 equal parts, and the length of the sine, tangent, or secant, &c. of any angle in such parts be given, the length of the sine, tangent, or secant of the same angle to any other given radius may be found by the common rule of proportion.

A table exhibiting the length of the sine, tangent, and secant of every degree and minute of the first quadrant in such parts, of which 10,000,000 make the radius, is called a *trigonometrical canon*; and it will always be, as the tabular radius is to any other given radius, so is the tabular sine, &c. of any angle to the sine, &c. of the same angle to the given radius.

26. THE chord of 60 degrees is equal to radius.

For then the angles ABC , BAC being equal (5. 1), each of them is 60 degrees = angle ACB ; therefore the triangle ACB being equiangular, is also equilateral (Cor. 6. 1); therefore the chord AB = radius AC .

27. The sine of 90 degrees, or a quadrant, or a right angle, is equal to radius. This is manifest from an inspection of the figure.

28. The tangent of 45 degrees is radius.

For then the angle ACT being half a right angle, the other acute angle ATC must also be half a right angle; therefore $AC=AT$ (6. 1).

29. The secant of 0 (or at the beginning of the circle) is radius.

30. The cosine of 0 degrees is radius, and the cosine of 90 degrees is 0.

31. The versed sine of 90 degrees is radius, and the versed sine of 180 degrees is the diameter.

32. Universally, the versed sine is always either the sum or the difference of the cosine and radius, namely, the sum in the two middle quadrants HD and DL, and the difference in the two extreme quadrants AH and LA.

33. As the arc increases from 90 degrees to 180 degrees, its sine, tangent, and secant decrease. Thus, as Ab increases, it is evident that the sine bf , the tangent At , and the secant Ct must decrease till the point b coincides with D. Consequently, if there be two arcs between 90° and 180° , the greater arc will have the less sine, tangent, and secant.

34. Let the arcs Ab and AB be supplements to each other, namely, Ab greater, and AB less than a quadrant; then will their sines bf and BF be equal.

Because $Ab+AB=180^\circ=Ab+bD$, the arc $AB=bD$, or the angle $bCD=BCA$. But the radii BC, bC are equal. Hence the right-angled triangles bCf, BCF are equal (26. 1), and $bf=BF$.

35. In like manner, Cf , the cosine of the arc Ab , is equal to CF , the cosine of the arc AB ; but is negative, because it falls on the other side of the centre C , whence the cosines have their origin.

36. Again, At the tangent, and Ct the secant of Ab , are respectively equal to AT the tangent, and CT the secant of AB .

The right-angled triangles TCA, tCA , having the angle BCA or TCA equal to the angle bCD or tCA , and CA common, are equal (26. 1), therefore $At=AT$, and $Ct=CT$. But the tangent and secant, being now produced in a contrary direction, will be negative.

37. In like manner, the sine, cosine, tangent, and secant of any arc terminating in the third quadrant DL, will be respectively the same as those of an arc equal to the excess of the

proposed arc above a semicircle. Thus, the sine of the arc $AHD\beta$ is βf , and is equal to BF , the sine of the arc $AB=DA$. And so of the other lines.

38. The sine, cosine, tangent, and secant of an arc terminating in the fourth quadrant LA , will be the same as those of an arc equal to the supplement of the proposed arc to a whole circle. Thus, the sine of the arc $AHDLE$, is EF , and is equal to BF , the sine of the arc $AB=AE$, the supplement of $AHDLE$ to a whole circle. And so of the other lines.

39. The versed sine Af of an arc Ab , above one quadrant, but under two quadrants, is equal to the difference between the versed sine of its supplement and the diameter; that is, $Af=AD-AF$. For $CF=Cf$, therefore $Af=DF=AD-AF$.

40. The versed sine of an arc above two quadrants is (not merely equal to, but) the same as the versed sine of its supplement to a whole circle.

Thus, the versed sine of the arc $AHD\beta$, and also of the arc $AEL\beta$ (or rather its equal $ABH\delta$) is Af : and the versed sine of the arc $AHDLE$, and of the arc $AB=AE$, is in both cases AF .

Scholium. From these propositions it follows, that a table of sines, tangents, secants, and versed sines, computed for every degree and minute of the first quadrant, will serve for the whole circle.

41. The right-angled triangles BCF , TCA , CKH , having the several acute angles BCF , TCA , CKH equal (29. 1), are equiangular; hence the following analogies are deduced.

1. Cosine CF : sine BF :: radius CA : tangent TA .
2. Cosine CF : radius CB :: radius $CA=CB$: secant CT .
3. Sine BF : radius CB :: radius $CH=CB$: cosecant CK .
4. Sine BF : cosine CF :: radius CH : cotangent HK .
5. Tangent TA : radius CA :: radius $CH=CA$: cot. HK .

Hence it appears that the radius is a mean proportional between the cosine and secant, between the sine and cosecant, and between the tangent and cotangent.

42. The tangents and cotangents of any two arcs of a circle are reciprocally proportional; that is, the tangent of the first arc : tangent of the second :: cot. of the second arc : cot. of the first.

Let T and C denote the tangent and cotangent of the first arc, t and c the tangent and cotangent of the second; then $T \times C = \text{radius}^2$ (41), and $t \times c = \text{radius}^2$; therefore $T \times C = t \times c$, therefore $T : t :: c : C$ (16. 6).

In the same manner it may be shown, that the cosines and secants of two arcs, and also the sines and cosecants, are reciprocally proportional.

43. The sine of any arc is equal to half the chord of double that arc.

For the radius CA, perpendicular to BE, bisects the chord BE in F (3. 3), and also the arc BAE subtended by it (26. 3), because the angle $BCA = ECA$; therefore BF, the sine of the arc BA, is half the chord BE, which subtends the arc BAE, the double of BA.

44. *Conversely.* The chord of any arc is double the sine of half that arc.

For the radius CA, perpendicular to the chord BE, bisects BE in F (3. 3), and also the arc BAE subtended by it (26. 3), because the angles ACB and ACE are equal; therefore the chord BE is double the sine BF of the arc AB=half the arc BAE.

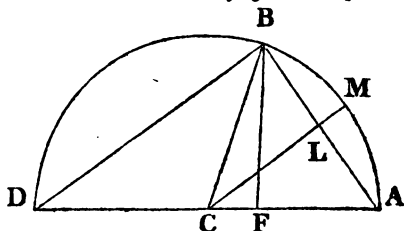
45. The sine of 30 degrees, the cosine of 60 degrees, and the versed sine of 60 degrees, are each equal to half the radius.

The chord of 60 degrees is equal to the radius (26), therefore the sine of 30 degrees, being equal to half the chord of 60 degrees (43), is equal to half the radius.

Again, the sine of an arc being equal to the cosine of its complement (23), the cosine of 60 degrees is equal to the sine of 30 degrees, and therefore is equal to half the radius.

Thirdly, the versed sine of an arc less than a quadrant being equal to the difference between the radius and the cosine, the versed sine of 60 degrees = radius — cosine of 60 degrees = radius — half radius = half radius.

46. On the diameter AD describe a semicircle ABD; draw AB the chord, and BF the sine, of the arc AB; draw the radius CLM perpendicular to the chord in L, and cutting the circle in M; then will the radius CLM bisect the chord AB in L (3. 3), and the arc AMB in M (26. 3). Hence AL will be the sine, and CL the cosine of the arc AM. Lastly, join the points D, B, then the triangle ABD will be right-angled at B (31. 3), and will be divided by BF, a perpendicular drawn from the right angle to the base, into two triangles AFB, DFB, similar to the triangle ABD, and to each



other (8. 6). Also, the two right-angled triangles ALC, ABD, having the acute angle LAC common, are similar to each other, and to the two triangles AFB, DFB.

Hence the following propositions are derived

1. The triangles DAB, BAF give the following analogy.

$$DA : BA :: BA : AF.$$

That is, the chord of any arc is a mean proportional between the diameter and the versed sine of that arc.

2. Again, the triangles AFB, DFB give the following analogy.

$$AF : BF :: BF : DF.$$

That is, the sine of any arc is a mean proportional between the versed sine and the covered sine.

3. From the similar triangles CAL, BAF, we deduce this analogy.

$$CA : AL :: AB \text{ or } 2 AL : AF.$$

That is, radius is to the sine of any arc AM, as twice that sine is to the versed sine of AMB double that arc.

4. Again, $CA : CL :: AB \text{ or } 2 AL : BF$.

That is, radius is to the cosine of any arc AM, as twice the sine of that arc is to the sine of AMB double that arc.

47. The following corollaries are deduced from art. 46.

Cor. 1. From no. 1, $DA \times AF = BA^2$ (16. 6).

That is, the rectangle under the diameter and the versed sine of any arc is equal to the square of the chord.

Cor. 2. From no. 1, $DA : AF :: DA^2 : BA^2 :: CA^2 : AL^2$.

That is, the diameter is to the versed sine of any arc, as the square of the radius is to the square of the right sine of half that arc.

Cor. 3. From no. 3, $2 AL^2 = CA \times AF$. But the radius CA is a constant quantity, therefore AL^2 varies as AF, that is, the square of the right sine of any arc is as the versed sine of double that arc.

Cor. 4. From no. 4, $CL \times 2 AL = CA \times BF$, or $CL \times AL = \frac{1}{2} CA \times BF$. Hence, if A denote any arc or angle, $\text{sine } A \times \cos. A = \frac{1}{2} \text{radius} \times \text{sine } 2 A$, that is, the rectangle under the sine and cosine of any angle is equal to the rectangle under half the radius and the sine of twice that angle, or, to half the rectangle under the radius and the sine of twice that angle.

* By the analytical properties of proportion. Or thus, $DA : AF :: DA \times DA \text{ or } DA^2 : DA \times AF = AB^2$ (15. 5).

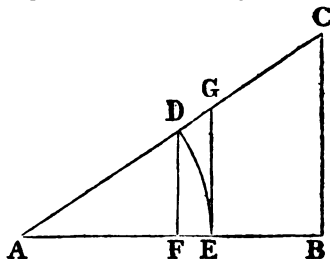
The Application of Trigonometry to the Mensuration of the Sides and Angles of Triangles described on a Plane, or Plane Trigonometry properly so called.

48. In any right-angled triangle ABC (fig. page 21), if the hypotenuse AC be made radius, the perpendicular BC becomes the sine, and the base AB the cosine of the angle at the base CAB. But if the base AB be made radius, the perpendicular BC becomes the tangent, and the hypotenuse AC the secant of the angle at the base CAB. This is manifest from the definitions.

PROP. I.

49. In any right-angled plane triangle, the hypotenuse is to either of the sides, as the radius to the sine of the angle opposite to that side, or to the cosine of the angle adjacent to that side.

Let ABC be a right-angled triangle, of which the hypotenuse is AC. From the centre A, with any radius AD, describe the arc DE; from D draw DF at right angles to AB, and from E draw EG touching the arc in E, and meeting AC in G. Then DF is the sine, and AF the cosine of the arc DE, or angle A; also EG is the tangent, and AG the secant of the arc DE, or angle A.



The two right-angled triangles AFD, ABC, having the angle C common, are equiangular; therefore $AC : CB :: AD : DF$, that is, $AC : CB ::$ the radius of the trigonometrical tables : sine of the angle A.

Again, $AC : AB :: AD : AF$, or $AC : AB :: R : \cos. A$ (cosine of the angle A).

PROP. II.

50. In any right-angled plane triangle, as either of the sides is to the other side, so is the radius to the tangent of the angle opposite to the latter. See last fig.

Because EG touches the arc in E, AEG is a right angle, and therefore is equal to the angle ABC. From the equiangular triangles ABC, AEG, $AB : BC :: AE :: EG$, or $AB : BC :: R : \tan. A$.

51. *Scholium to Prop. I, II.*

If the lengths of AC and CB, or AC and AB, or AB and CB, be known in feet, inches, or any other measure, the angles may be found by the following proportions.

$$AC : AB :: R :: \sin C.$$

Whence the sine of the angle C is known, and consequently the quantity of the angle C may be found by seeking the sine in a table of sines and tangents.

$$\text{Again, } AC : CB :: R : \sin A,$$

$$\text{and } AB : BC :: R : \tan. A,$$

$$\text{and } BC : AB :: R : \tan. C,$$

$$\text{and } AC : AB :: R : \cos. A.$$

Whence the tangents and sines (and consequently the angles) are discovered by the rule of proportion.

We may invert these analogies, as follows:

$$\sin C : R :: AB : AC,$$

$$\sin A : R :: CB : AC,$$

$$\tan. A : R :: BC : AB,$$

$$\tan. C : R :: AB : BC,$$

$$\cos. A : R :: AB : AC.$$

The first three terms of each of these analogies are supposed to be known; hence the fourth term is found.

52. Cor. If these analogies be arithmetically expressed, and radius be supposed =1; then $\sin A = \frac{BC}{AC}$, $\cos. A = \frac{AB}{AC}$, $\tan. A = \frac{BC}{AB}$.

PROP. III.

53. In every triangle, if a perpendicular be drawn from any of the angles on the opposite side, the segments of that side will be to each other as the tangents of the parts into which the vertical angle is divided by the perpendicular, or as the cotangents of the angles at the base.

If AD be drawn perpendicular to the base BC, it will divide the triangle ABC into two right-angled triangles CAD, BAD. About the centre A, with the radius AD, suppose arcs to be described on each side of the point D, then DC, DB will be the tangents of the opposite angles CAD, BAD. Hence

$$AD : R :: DC : \tan. CAD \text{ (50),}$$

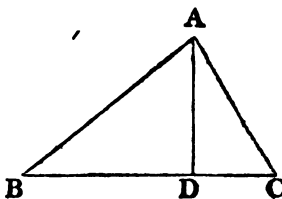
$$\text{and } AD : R :: DB : \tan. BAD.$$

consequently $DC : \tan. CAD :: DB : \tan. BAD \text{ (11. 5),}$

$$\text{or } DC : DB :: \tan. CAD : \tan. BAD \text{ (16. 5),}$$

$$\text{or } DC : DB :: \cot. C : \cot. B,$$

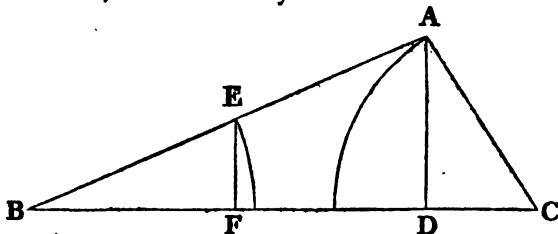
because the tangent of an arc or angle is equal to the cotangent of its complement (23).



PROP. IV.

54. The sides of any triangle are to one another as the sines of their opposite angles; and, conversely, the sines of the angles of any triangle are to one another as the sides which are opposite to the angles.

In the triangle ABC, the side AB : side AC :: sine $\angle ACB$, opposite to the former side AB : sine $\angle ABC$, opposite to the latter side AC; and conversely.



On the side BA (produced if necessary) take $BE = CA$, and from the points A, E let fall the perpendiculars AD, EF upon BC; then will AD and EF be the sines of the angles ACB and ABC to the equal radii CA and BE.

The right-angled triangles ADB, EFB, having the acute angle at B common, are equiangular; therefore $AB : EB$ or $AB : AC :: AD : EF$, that is, $AB : AC :: \sin. \angle C : \sin. \angle B$.

Otherwise.

See fig. Prop. 3. From any angle A of the triangle ABC draw AD perpendicular to BC. About the centres B and C, with the radii BA and CA, suppose arcs to be described; then AD will be the sine of the angles B and C.

In the right-angled triangle ABD,

$$AB : AD :: R : s. B \text{ (49),}$$

and in the right-angled triangle ACD,

$$AC : AD :: R : s. C.$$

$$\text{Hence } s. B \times AB = R \times AD = s. C \times AC.$$

$$\text{Therefore } AB : AC :: s. C : s. B \text{ (16. 6).}$$

$$\text{In the same manner, } AB : BC :: s. C : s. A.$$

55. Cor. 1. Hence, in any triangle, if two sides and an angle opposite to one of them be given, the remaining side and angles may be found; or, if two angles and a side opposite to one of them be given, the other angle and sides may be found.

Thus, if the sides AC, AB, and the angle C, be given, to find the rest, then

$$AB : s. C :: AC : s. B.$$

Because the sum of the angles B and C is the supplement of A (32. 1), $180^\circ - (\angle C + \angle B) = \angle A$.

$$\text{Whence, sine } C : AB :: s. A : BC.$$

Again, if the angles A and C, and the side AB, be given, to find the rest, then

$$\text{Sine } C : AB :: s. A : BC,$$

$$\text{and } 180^\circ - (\angle A + \angle C) = \angle B.$$

$$\text{Hence, sine } C : AB :: s. B : AC.$$

56. Cor. 2. If all the angles of a triangle be given, the proportion of the sides to one another may be found, but not the real lengths of the sides. For the proportion of the sides to one another is the same as that of the sines of the angles which are opposite to them.

LEMMA III.

57. If the semi-sum and semi-difference of any two quantities be added together, the aggregate will be the greater quantity; and if the semi-difference be subtracted from the semi-sum, the remainder will be the less quantity.

This proposition is demonstrated by the writers on Algebra, and may be proved geometrically as follows.

Let AB , BC be two unequal magnitudes, of which AB is the greater. Place AB , BC in the same straight line AC ; bisect AC in D , and take $AE=BC$. Then AC is the sum, and EB is the difference of the two magnitudes AB , BC .

$A \text{-----} B$ Because AD is equal to DC , and
 $B \text{-----} C$ AE to BC , $AD-AE$ is equal to
 ----- $DC-BC$, that is, $DE=DB$; there-
 $A \quad E \quad D \quad B \quad C$ fore DE or DB is half the differ-
 ence of the magnitudes AB , BC . But $AB=BD+DA$, that is,
 to half the difference added to half the sum; and $BC=$
 $DC-DB$, that is, to the excess of half the sum above half
 the difference.

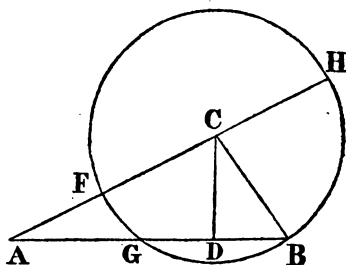
58. Cor. If the semi-sum be subtracted from the greater quantity, the remainder will be the semi-difference. $AB-AD=BD$.

PROP. V.

59. In any triangle, if a perpendicular from the vertex to the base fall within the triangle, the base is to the sum of the other two sides, as their difference is to the difference of the segments of the base made by the perpendicular.

Let ABC be the proposed triangle, C the vertex, AB the base, CD the perpendicular dividing the base into the segments AD , DB . About the centre C , with the radius CB , the less of the two sides, describe a circle cutting the base in G , the side AC in F , and AC produced in H ; then will

$AH=AC+CH=AC+CB$, be the sum of the sides, and $AF=AC-CF=AC-CB$, be the difference of the sides. Because $DB=DG$ (3. 3), $AG=AD-DB$ is the difference of the segments of the base. Hence $AB \times AG = AH \times AF$ (Cor. 36. 3), therefore $AB : AH :: AF : AG$ (16. 6); that is,
 $AB : AC+CB :: AC-CB : AD-DB$.

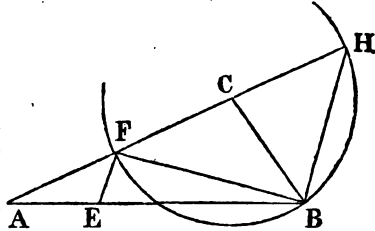


60. *Scholium.* In any triangle ABC , if a perpendicular be let fall from the greatest angle C , it will fall within the triangle, and will divide it into two right-angled triangles ACD , BCD , whose hypotenuses AC , BC are the sides of the proposed triangle, and whose bases AD , DB are the segments of its base. Now if all the three sides of the triangle ABC be given, then the base AB , or the sum of the segments AD , DB , is given, and the difference of the segments may be found by the proposition. But if the sum and difference of the segments be known, the segments may be found by article 57. Thus, in the two right-angled triangles ACD , BCD , the hypotenuses AC , BC are given, and the bases AD , BD are found by the proposition; whence the angles at the base CAD , CBD may be found by art. 49, and consequently the angle ACB , which is the supplement of the angles at A and B , may be found by Cor. 32. 1.

PROP VI.

61. In any triangle, the sum of any two sides is to their difference, as the tangent of half the sum of the opposite angles is to the tangent of half their difference.

Let ABC be the proposed triangle, whose sides are AC , BC , and base AB . About the centre C , with the radius CB , the less of the two sides, describe a circle cutting the longer side AC in F , and AC produced in H ; then is AH the sum of the sides AC and CB , and AF their difference. Draw FB and HB , then is the angle HCB the sum of the angles at the base (32. 1), and the angle HFB = half the angle HCB (20. 3), half the sum of the angles at the base.



The triangle FCB is isosceles, therefore the angle $CFB = CBF$ (5. 1), therefore the angle CBF is half the sum of the angles at the base. But $ABF = CBA - CBF$, that is, the angle ABF is the difference between the greater of the angles at the base and their semi-sum, therefore ABF is the semi-difference of the angles at the base (58).

Draw FE parallel to BH , then the angle $EFB = HBF$ (29. 1) = a right angle (31. 3). Therefore if F be made the centre, and FB the radius, HB will be the tangent of the angle HFB , the semi-sum of the angles at the base; and if B

be the centre, with the same radius FB, then FE will be the tangent of the angle EBF, the semi-difference of the angles at the base.

The triangles AFE, AHB, having the angles AFE and AHB equal (29. 1), and the angle at A common, are similar. Therefore $AH : AF :: HB : FE$, or $AC + CB : AC - CB :: HB : FE$; that is, the sum of the sides : difference of the sides :: tan. of half the sum of the angles at the base : tan. of half their difference.

62. *Scholium.* If two sides AC, BC of any triangle ABC, and the angle ACB included between them, be given, the other side and angles may be found.

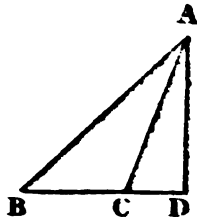
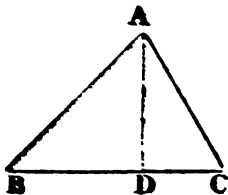
For, from the angle ACB we can find its supplement to 180° (18. 1), equal to the angle HCB, half of which is the semi-sum of the angles at the base; and their semi-difference is found by this proposition. Then, the semi-sum and semi-difference of the angles being known, the angles themselves, CBA and CAB, may be found by art. 57. Two sides and all the angles of the proposed triangle being now known, the third side AB may be found by art. 54, as follows. Since $\angle A : BC :: \angle CACB : AB$.

Note. The preceding propositions are sufficient for the resolution of all the cases of rectilinear triangles. The following proposition is added, because it is useful in many geometrical investigations, and in the solution of triangles in certain circumstances.

PROP. VII.

65. In any rectilinear triangle, twice the rectangle contained by any two sides is to the difference between the sum of the squares of those sides and the square of the base, as the radius to the cosine of the angle contained by the two sides.

Let ABC be any triangle; $2 AB \cdot BC$: difference between $AB^2 + BC^2$ and $AC^2 :: R : \cos. B$.



From A draw AD perpendicular to BC, then the difference between $AB^2 + BC^2$ and AC^2 is equal to $2 BC \cdot BD$ (13. 2). In the right-angled triangle ABD, $BA : BD :: R : \cos. B$ (49), therefore $2 BC \cdot BA : 2 BC \cdot BD :: R : \cos. B$ (15. 5), that is, $2 AB \cdot BC : \text{difference between } AB^2 + BC^2 \text{ and } AC^2 :: R : \cos. B$.

64. Cor. 1. If radius=1, then $BD = BA \times \cos. B$ (49), therefore $2 BC \cdot BA \times \cos. B = 2 BC \cdot BD$; therefore when B is acute, $2 BC \cdot BA \times \cos. B = BC^2 + BA^2 - AC^2$ (13. 2), therefore $AC^2 + 2 \cos. B \times BC \cdot BA = BC^2 + BA^2$, therefore $AC^2 = BC^2 - 2 \cos. B \times BC \cdot BA + BA^2$, therefore $AC = \sqrt{(BC^2 - 2 \cos. B \times BC \cdot BA + BA^2)}$.

If B be an obtuse angle, it may be proved in the same manner, that $AC = \sqrt{(BC^2 + 2 \cos. B \times BC \cdot BA + BA^2)}$.

65. Cor. 2. Because $AC^2 = AB^2 + BC^2 - 2 \cos. B \times AB \cdot BC$, $2 \cos. B \times AB \cdot BC = AB^2 + BC^2 - AC^2 = AB^2 + (BC + AC) \times (BC - AC)$, therefore $\cos. B = \frac{AB^2 + (BC + AC) \times (BC - AC)}{2 AB \cdot BC}$.

66. *Scholium.* When two sides and the included angle of any plane triangle are given, the third side may be found from art. 64. This method of finding the third side of a triangle is very useful in many geometrical investigations, but is not adapted to computation by logarithms, because the quantity under the radical sign cannot be resolved into simple multipliers.

When the three sides of any plane triangle are given, the angles may be found immediately from art. 65. When the sides are expressed by great numbers it will be more convenient to find the difference of the segments of the base, by prop. 5, and then the angles at the base, by prop. 1.

SECTION II.

RULES OF TRIGONOMETRICAL CALCULATION.

67. THE general problem which trigonometry proposes to resolve is this. *In any plane triangle, of the three sides and the three angles, three parts being given, one of which is a side, to find any of the other three parts.*

The parts of a triangle which are said to be given, are understood to be expressed by numbers, or by their numerical

values; the angles in degrees, minutes, &c.; and the sides in feet, or any other known measure.

The reader must remember, that the sines, cosines, &c. which are used in the calculations in this section, are the *numerical measures* of the geometrical lines bearing the same names, the measure of the radius of the circle being supposed unity.

The problem is restricted to those cases in which one side at least is given, because, if the three angles only be given, the magnitudes of the sides cannot be determined, but only the *ratios* of their magnitudes. Innumerable triangles, equiangular to one another, may exist; but the sides of none of them may be equal to the sides of another, though the ratios of the sides to one another will be the same in all the triangles (4. 6). Therefore, if the three angles only of a triangle be given, we can determine nothing but the ratios of the sides, which are the same as the ratios of the sines of the opposite angle (56).

All the *varieties* which can happen in the solution of plane triangles are comprised in three cases, as follows.

1. When two of the three given things are a side and its opposite angle.

2. When two sides and their included angle are given.

3. When the three sides are given.

For the conveniency of calculation it is usual to divide the general problem into two problems, according as the proposed triangle is right-angled or oblique-angled.

A right-angled triangle consists of three *variable* parts, the three sides and the two acute angles. The right angle being a constant quantity is not reckoned. An oblique-angled triangle consists of six variable parts, the three sides and the three angles.

PROBLEM I.

68. In a right-angled triangle, of the three sides and the three angles, two being given, besides the right angle, and one of them being a side, it is required to find the other three parts of the triangle.

If one angle of a triangle be right, the other two angles are acute, and together make a right angle (32. 1). Therefore when one of the acute angles of a right-angled triangle is given, the other also is given, being its complement to a right angle, or 90 degrees.

The sine of either of the acute angles is the cosine of the other (23), and therefore may be used instead of the other, whenever it renders the operation more simple.

In the calculation of right-angled plane triangles, any side, whether given or required, may be made radius to find a side; but a *given* side must always be made radius to find an angle. Hence, to find an angle, begin the proportion with a side opposite to a *given* angle; and to find a side, begin the proportion with an angle opposite to a *given* side.

69. This problem contains four particular cases, and the solutions, or rules of calculation, all depend upon the first and second propositions.

Case 1. When the hypotenuse and one of the acute angles are given, to find the rest.

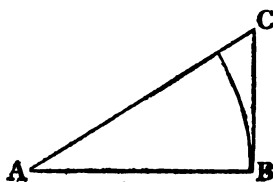
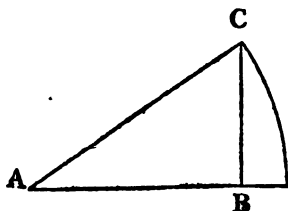
2. When a side and one of the acute angles are given.

3. When the hypotenuse and one of the sides are given.

4. When the two sides are given.

In the following table the first column contains the things given, the second contains the things required, and the third contains the rules or proportions by which they are found.

GIVEN	SOUGHT	SOLUTION
I. AC & C, the hyp. & an angle.	AB	$R : s. C :: AC : AB$
	BC	$R : \cos. C :: AC : BC$
II. AB & A, a side and one of the acute angles.	AC	$\cos. A : R :: AB : AC$
	BC	$R : \tan. A :: AB : BC$
III. AC & BC, the hyp. and a side.	A	$AC : BC :: R : s. A$
	AB	$R : \cos. A :: AC : AB$
IV. AB & BC, the two sides.	A	$AB : BC :: R : \tan. A$
	AC	$\cos. A : R :: AB : AC$



Remarks on the Solutions in the table.

70. In the second case, when the base AB and the angle A are given, to find the hypotenuse AC, a solution may also be obtained by help of the secant. For $AB : AC :: R : \sec. A$, therefore $R : \sec. A :: AB : AC$.

In the third case, when the hypotenuse AC and the side BC are given, to find AB, a solution may be obtained from 47. 1. For $AB^2 = AC^2 - BC^2$, therefore $AB = \sqrt{AC^2 - BC^2}$. This value of AB may be easily calculated by logarithms, if the quantity $AC^2 - BC^2$ be resolved into two factors. Thus, $AC^2 - BC^2 = (AC + BC) \times (AC - BC)$ (Cor. 5. 2); therefore $AB = \sqrt{(AC + BC) \times (AC - BC)}$. Hence, $\log. AB = \frac{\log. (AC + BC) + \log. (AC - BC)}{2}$.

In the fourth case, when AB and BC are given, AC may be found from 47. 1. For $AC = \sqrt{AB^2 + BC^2}$. But $AB^2 + BC^2$ cannot be separated into multipliers; therefore when AB and BC are represented by great numbers, this rule is not convenient for computation by logarithms. It is then best to seek first the tangent of A, by the analogy in the table, $AB : BC :: R : \tan. A$. But if the angle A be not required, it is sufficient, after $\tan. A$ is found by this proportion, to take from the trigonometrical tables the cosine which corresponds to $\tan. A$, and then to compute AC by this proportion, $\cos. A :: R :: AB : AC$.

71. If the two acute angles of a right-angled triangle be denoted by A, C, and their opposite sides by a, c respectively, and the hypotenuse by b; then prop. I, II may be expressed by general equations, by means of which all the cases of right-angled triangles may be easily resolved.

$$1. \frac{a}{r} = \frac{b \sin A}{r} = \frac{b \cos C}{r},$$

$$\text{or } \log. a = \log. b + \log. \sin A - \log. r = \log. b + \log. \cos C - \log. r.$$

$$2. \frac{c}{r} = \frac{a \tan A}{r}, \text{ or } \log. c = \log. a + \log. \tan A - \log. r.$$

If any two of these three parts (beside radius) be given, the third part may be found from these equations.

PROBLEM II.

72. In any oblique-angled triangle, of the three sides and the three angles three parts being given, and one of these being a side, it is required to find the other three parts.

This problem contains four cases, and the solutions of them depend upon the preceding propositions.

Case 1. When two angles and a side are given, to find the rest.

2. When two sides and an angle opposite to one of them are given.

3. When two sides and the included angle are given.

4. When the three sides are given.

73. The following properties of plane triangles are requisite in the solution of the cases of oblique-angled triangles.

The sum of the three angles of every plane triangle is equal to two right angles, or 180 degrees (32. 1). Hence

1. When two angles of a triangle are given, the third angle is said to be given; for it is the supplement of the other two angles, and is found by subtracting their sum from 180° .

2. When one angle of a triangle is given, the sum of the other two angles may be found by subtracting the given angle from 180° .

The angles opposite to the two least sides of a plane triangle are acute; and if there be an obtuse angle, it is opposite to the greatest side.

A perpendicular drawn from the opposite angle to the longest side will fall within the triangle; and the greater segment will be next to the greater side, and the less segment to the less side.

If both the angles at the base be acute, the perpendicular will fall within the triangle; but if one of them be obtuse, the perpendicular will fall without the triangle, beyond the obtuse angle; and in both cases the greater segment will be next to the greater side, and the less segment to the less side.

74. *Solution of the Cases of Oblique-angled Triangles.*

Case 1. Given two angles A and B, and one side AB, to find the other two sides. See fig. art. 63.

Solution. Because the angles A, B are given, the angle C is also given, being the supplement of A+B (73). Then

$$\begin{aligned} \text{sine } C : \text{s. } A &:: AB : BC \text{ (54),} \\ \text{and sine } C : \text{s. } B &:: AB : AC. \end{aligned}$$

Case 2. Given two sides AB, AC, and the angle B opposite to one of them, to find the other angles A, C, and the third side BC.

Solution. $AC : AB :: \text{s. } B : \text{s. } C$ (54). Hence $\angle A = 180^\circ - B - C$. Then, $\text{s. } B : \text{s. } A :: AC : CB$.

Case 3. Given two sides AB, AC, and the angle A contained between them, to find the other angles B and C, and the third side BC.

Solution. $AB + AC : AB - AC :: \tan. \frac{1}{2} (B + C) : \tan. \frac{1}{2} (B - C)$ (61). Since $\frac{1}{2} (B + C)$ and $\frac{1}{2} (B - C)$ are given, the angles B and C may be found. For $B = \frac{1}{2} (B + C) + \frac{1}{2} (B - C)$ (57), and $C = \frac{1}{2} (B + C) - \frac{1}{2} (B - C)$. Then $\text{s. } B : \text{s. } A :: AC : BC$.

Remark. Instead of the tangent of half the sum of the two unknown angles we may take the cotangent of half the given angle, or the tangent of half its supplement; for these three tangents are equal to one another. Thus, the sum of the angles B and C is less than two right angles (32. 1), therefore half their sum is less than one right angle, therefore $\tan. \frac{1}{2} (B + C) = \cot. \frac{1}{2}$ their complement A (23). Also, $180^\circ - A = B + C$. Hence the first analogy becomes, $AB + AC : AB - AC :: \cot. \frac{1}{2} A$ or $\tan. \frac{1}{2} (180^\circ - A) : \tan. \frac{1}{2} (B - C)$.

Otherwise. Given AB, BC, and the included angle B, to find the rest.

Upon one of the given sides, as BC, let fall a perpendicular AD from the opposite angle A. Then, in the right-angled triangle ABD the hyp. AB and all the angles are given; therefore $R : \cos. B :: AB : BD$ (49). Now BD being found, and BC being given, DC will be given. Hence $BD : DC :: \tan. \text{given angle DAB} : \tan. DAC$ (53). Hence $BAC = BAD + DAC$, when the perp. falls within the triangle, and $BAC = BAD - DAC$, when the perp. falls without. Hence $C = 180^\circ - B - BAC$. Lastly, $s. C : s. B :: AB : AC$.

Remark. When two sides and the included angle of a plane triangle are given, the third side may be found by art. 64, without first finding the angles. But this method of finding the third side is not adapted to calculation by logarithms, because the quantity under the radical sign cannot be resolved into simple factors. Therefore the operation by this rule will be tedious, unless the two given sides be expressed by small numbers. When the two given sides are expressed by great numbers, it will be more convenient to find the angles and the third side by either of the preceding methods.

Case 4. Given all the sides, to find the angles.

Solution. Make the longest side the base, and let fall a perp. upon it from the opposite angle. The angles B and C, adjacent to the longest side, will be acute. Let F denote the difference of the segments of the base. Then $BC : BA + AC :: BA - AC : F$ (59). Hence $\frac{1}{2} (BC + F) =$ greater segment of the base (57), and $\frac{1}{2} (BC - F) =$ less segment. Then $CA : CD :: R : \cos. C$, and $BA : BD :: R : \cos. B$. Hence $A = 180^\circ - B - C$.

When the sides are not expressed by great numbers, the angles may be easily found by art. 65.

$$\cos. B = \frac{AB + (BC + AC) \times (BC - AC)}{2 AB \cdot BC}$$

It may always be known whether the angle sought is greater or less than a right angle; for if the square of the side opposite to it be greater than the sum of the squares of the other two sides, it is an obtuse angle (12. 2); but if less, it is an acute angle (13. 2). Hence this case is not ambiguous.

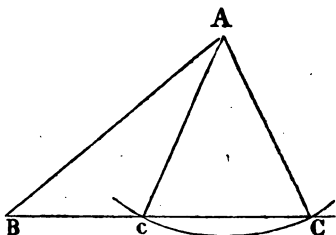
75. Remark. The angle C , in case 2, may have two values, one greater, the other less than a right angle. For an angle and its supplement having the same sine (34), the angle belonging to the sine of the angle C found by the analogy, may be either that which is found in the tables, or the supplement of it.

This ambiguity does not arise from any defect in the solution of the problem, but from a circumstance essential to the problem; for when AC , the side opposite to the given angle B , is less than the other given side AB , there are two triangles, each of which has the sides AB , AC , and the angle B of the same magnitude; but yet these two triangles are not equal, the angle opposite to AB in one triangle being the supplement of the angle opposite to AB in the other. The truth of this appears as follows.

From the centre A , with the radius AC , describe an arc cutting BC in c , and join A, c . Then ACc is an isosceles triangle, and therefore the angles ACc and AcC are equal. Therefore the angle AcB , which is the supplement of AcC , is also the supplement of ACB . Now ACB , AcB are the angles found by the analogy in the solution. But the sine found by the analogy may be the sine of either of the angles ACB , AcB (34); and either of these angles, one of which will be acute, and the other obtuse, may be the angle opposite to AB in the proposed triangle.

Again, the triangles ABC , ABc have the side AB and the angle B common, and the sides AC , Ac equal; but they have not the remaining sides equal, and the remaining angles equal, namely, $BC=Bc$, the angle $BcA=BCA$, and $BAC=BAC$. Consequently the triangles ABC , ABc are not equal.

Thus, when the given side AB is greater than AC , and consequently the angle C greater than B , there are two triangles which satisfy the conditions of the problem. But when AC is greater than AB , the intersections C and c fall on opposite sides of the angle B , and therefore the two triangles have



not the angle B common to both. In this case the angle required being necessarily less than B is an acute angle, and therefore the solution ceases to be ambiguous.

From the two angles ACB, AcB the two angles BAC, BA_cC may be found. The angle BAC is the supplement of the sum of the angles B and C (32. 1), therefore the sine of BAC is the same as the sine of B+C. The angle BA_cC=AcC—B (32. 1)=ACc—B. Hence, the angle C being found, sine C : s. (C+B) :: AB : BC, and s. C : s. (C—B) :: AB : BC.

76. If the three angles of an oblique triangle be denoted by A, B, C, and their opposite sides by a, b, c, then propositions IV, V, VI, VII, may be expressed by general equations, by means of which all the cases of plane triangles may be resolved.

1. Sine $A = \frac{a \text{ s. } B}{b}$, or $\log. s. A = \log. a + \log. s. B - \log. b$.
2. Tan. $\frac{A \text{ } B}{2} = \frac{a \text{ } b}{a + b} \times \tan. \frac{A+B}{2} = \frac{a \text{ } b}{a + b} \times \cot. \frac{1}{2} C$,
or $\log. \tan. \frac{1}{2} (A \text{ } B) = \log. (a \text{ } b) - \log. (a + b) + \log. \cot. \frac{1}{2} C$.
3. $BD \text{ } DC = \frac{(a+b) \times (a \text{ } b)}{c}$,
or $\log. (BD \text{ } DC) = \log. (a+b) + \log. (a \text{ } b) - \log. c$.
4. Cos. $A = r \times \left(\frac{b^2 + c^2 \text{ } a^2}{2 bc} \right)$.

In this equation, when the angle A is acute, $b^2 + c^2$ will be greater than a^2 ; and when A is obtuse, $b^2 + c^2$ will be less than a^2 .

If any three of these four parts (beside radius) be given, the fourth part may be found from these equations.

Observations preparatory to the Practical Solution of the Cases of Plane Triangles.

77. Logarithms are a set of artificial numbers adapted to the common or natural numbers, 1, 2, 3, 4, 5, &c. for the purpose of facilitating arithmetical calculations. The addition and subtraction of logarithms correspond to the multiplication and division of common numbers. Therefore logarithms are generally used in trigonometrical and astronomical calculations, to avoid the tedious operations of multiplication and division in finding a fourth proportional to three given numbers.

78. Before the learner attempts the solution of the following cases of triangles he ought to know decimal fractions, practical geometry, and the use of the tables of the logarithms of

numbers, and of sines and tangents. The construction and use of these tables are generally prefixed to them in the collections of mathematical tables, of which the best are those of Taylor, 4to, Callet, 8vo, Hutton, 8vo, Mackay, 8vo, and the tables commonly called the Requisite Tables, 8vo.*

79. The sines, tangents, &c. are also called the *natural* sines, tangents, &c. of the arcs or angles to which they belong; and the logarithms of the numbers by which they are represented, are called the logarithmic sines, tangents, &c. As one or other of these lines makes a part of every trigonometrical operation, they have been calculated to a given radius, for every degree, minute, and (sometimes) second of the quadrant, and ranged in tables for practical use. Hence, by means of such tables, the sine, tangent, &c. of any arc or angle may be found by inspection; and, on the contrary, the arc or angle to which any sine, tangent, &c. belongs, may also be found by inspection.

80. Upon a table of sines and tangents, and the doctrine of similar triangles, depends the practical solution of the several cases of plane trigonometry, which may be performed either by the natural or the logarithmic sines, tangents, &c., as occasion requires. The logarithmic sines and tangents are commonly used, because the calculations by them are performed by addition and subtraction; but the natural sines and tangents require the more tedious operations of multiplication and division.

81. Each of the cases of rectilinear triangles admits three different methods of solution. 1. By geometrical construction. 2. By arithmetical calculation. 3. Instrumentally.

In the first method, the triangle is constructed by laying down the sides from a scale of equal parts, and the angles from a scale of chords, or a protractor. The unknown parts of the triangle thus constructed are found by measuring them on the same scale or instrument from which the known parts were taken.

In the second method let the analogy be formed according to the proper rule above delivered; then, if the natural numbers be used, multiply the second and third terms together, and divide the product by the first; the quotient will be the fourth term required. If logarithms be used, add the logarithms of the second and third terms, and from the sum sub-

* F. Nichols proposes to reprint De La Lande's Stereotype Tables of the Logarithms of Numbers, Sines and Tangents, with the addition of other useful tables, in one volume 18mo. Designed chiefly for the use of students in the Universities.

tract the logarithm of the first; the remainder will be the logarithm of the fourth term; and the number answering to that logarithm in the tables will be the number sought.

In the third method, where the rule commonly called Gunter's Scale is used, form the analogy, as in the last method; then extend the compasses, on the logarithmic lines described on one side of the scale, from the first term to the second or third, as they happen to be of the same name; that extent will reach from the other term to the fourth term required, both extents being directed toward the same end of the scale.

The second method, in which the operation is performed by logarithms, is generally practised. The other two methods are chiefly of use as checks on the arithmetical calculations, or in certain simple cases, where approximate values of the quantities sought are deemed sufficient; but must not be applied when accuracy is required.

82. In any operation, when one or more logarithms are to be subtracted from the others, it will be sometimes more convenient to take their *complements* (or what each logarithm wants of 10.0000000) instead of the logarithms, and then to add all the logarithmic terms together, and to subtract from the index of their sum as many times 10 as there were logarithms to be subtracted. Thus, if the log. to be subtracted be 3.4932758, its complement 6.5067242 may be added; and if the log. to be subtracted be 9.07432600, its complement 0.92567400 may be added.

If the index of the logarithm, whose complement is to be taken, be greater than 10, subtract the logarithm from 20, and the remainder will be its complement. Thus, the complement of the log. 12.4907327 is 7.5092673. If this complement be added to another logarithm, 20 must be subtracted from the index of the sum.

If the logarithm of a decimal is to be subtracted, add 10 to the index (which is negative), and take the complement of the rest of the figures, as before. Thus, the complement of the log. $\bar{3}.5972648$ is 2.4027352.

83. In trigonometrical calculations the sine, cosine, tangent, &c. of the same angle often occur. Therefore let all the analogies be formed (if there be more than one), before you begin the numerical operations, and leave the terms blank, that they may be afterward filled up with their proper logarithms. You will thus perceive what angles are repeated in the analogies, and therefore can, at one opening of the tables, take out all the logarithms belonging to the same angle, and write them in

their proper places. By this means you will prevent the loss of time and labour, which would happen by taking out the logarithms of the angles separately.

84. In the subsequent calculations the angles are mostly found to the nearest minute, as they appear in the common tables of logarithms; which is sufficient for almost all purposes. The seconds, if required, may be found by the rules prefixed to tables of logarithms. In Taylor's Tables the sines, &c. extend to seconds through the quadrant; and degrees, minutes, and seconds can be found at once by inspection. See *Appendix*.

85. *Practical Solution of the Cases of Right-angled Triangles.*

Fig. page 21.

Case 1. In any right-angled triangle ABC, given the hypotenuse AC 324, and the angle at the base BAC $48^{\circ} 17'$, to find the other parts of the triangle.

*By Construction.**

Draw an indefinite line AC. From the point A, with the chord of 60° as radius, describe an arc, upon which lay off the angle $A=48^{\circ} 17'$, taken from the same line of chords. From some convenient scale of equal parts lay down $AC=324$. From A, through the extremity of the arc which measures the given angle, draw an indefinite line AB; and from C let fall a perpendicular CB upon AB. Measure the sides AB, BC on the same scale from which AC was taken, and they will be found to be 215 and 242 nearly.

The angle C, being the complement of A, is $90^{\circ}-48^{\circ} 17'=41^{\circ} 43'$.

* The learner must construct the following problems on paper, according to the directions given for the construction of each. To save expense, the actual construction of the figures is omitted in this work; but the rules for constructing them will be sufficiently illustrated by inspecting the triangles in page 18 and 21. The construction of most of the examples is so plain and simple, that it is not necessary to give particular rules for all the cases of rectilinear triangles.

By Calculation.

The angle $C = 90^\circ - 48^\circ 17' = 41^\circ 43'$.

Then to find AB, BC.

Radius : sine $\angle A :: AC : CB$.

Radius } : sine $48^\circ 17' :: 324 : CB$.
or sine 90

Radius : sine $\angle C :: AC : AB$.

Radius : sine $41^\circ 43' :: 324 : AB$.

Hence $BC = AC \times \text{sine } A \div R$, and $AB = AC \times \text{sine } C \div R$.

Solution by Natural Sines.

$1 : .7464446 :: 324 : BC = 241.8480504$.

$1 : .6654475 :: 324 : AB = 215.60499$.

Solution by Logarithmic Sines.

Log. $BC = \text{log. sine } A + \text{log. } AC - \text{log. Radius}$.

Log. sine $A \ 48^\circ 17' \quad - \quad - \quad - \quad 9.8729976$

Log. $AC \ 324 \quad - \quad - \quad - \quad 2.5105450$

Sum $- \quad 12.3835426$

Log. Radius $- \quad - \quad - \quad 10.0000000$

Log. $BC \quad - \quad - \quad - \quad 2.3835426$

Log. $AB = \text{log. sine } C + \text{log. } AC - \text{log. Radius}$.

Log. sine $41^\circ 43' \quad - \quad - \quad 9.8231138$

Log. $324 \quad - \quad - \quad - \quad 2.5105450$

Sum $- \quad 12.3336588$

Log. Radius $- \quad - \quad - \quad 10.0000000$

Log. $AB \quad - \quad - \quad - \quad 2.3336588$

Whence $BC = 241.84805$, and $AB = 215.605$.

The solution of the cases of triangles by logarithmic sines, &c. is generally more expeditious than that by natural sines, &c., and therefore is preferable in practice. Besides, some of the best tables do not contain natural sines, but all contain logarithmic sines, which are indispensable in trigonometrical calculations.

Instrumentally.

In the first proportion, extend the compasses from 90° to $48^\circ 17'$ on the line of sines. That extent will reach from 324 to 242 on the line of numbers, and will be the length of BC.

In the second proportion, extend the compasses from 90° to $41^\circ 43'$ on the line of sines. That extent will reach from 324 to 215.6 on the line of numbers, and will be the length of AB.

Note. The radius is 90° of sines (27), and 45° of tangents (28), and 0 of cosines (30) and secants (29).

Ex. 2. Given the hypotenuse AC 121 yards, and the angle at the base A $55^\circ 30'$, to find the rest.

Answer. BC=99.719 yards, AB=68.535, angle C= $34^\circ 30'$.

Case 2. Given the side AB 125, and the angle A $51^\circ 19'$, to find the rest.

By Construction.

Draw an indefinite line AB, and from A to B set off 125, from a scale of equal parts.

Make the angle A= $51^\circ 19'$, as in case 1. At the point B erect a perpendicular, and from A through the extremity of the arc which measures the angle A draw AC meeting the perpendicular in C. Then AC, BC, measured on the same scale of equal parts, will be 199.5 and 156. The angle C= $90^\circ - 51^\circ 19' = 38^\circ 41'$.

By Calculation.

Angle C= $90^\circ - 51^\circ 19' = 38^\circ 41'$:

Radius : tan. A :: AB : BC.

R : tan. $51^\circ 19'$:: 125 : BC.

Cos. A or sine C : R :: AB : AC.

Cos. $51^\circ 19'$ or sine $38^\circ 41'$: R :: 125 : AC.

or, R : sec. A :: AB : AC.

R : sec. $51^\circ 19'$:: 125 : AC.

Hence $BC = AB \times \tan. A \div R$, and $AC = AB \times R \div \cos. A = AB \times R \div \sin C = AB \times \sec. A \div R$.

*Solution by Natural Tangents and Secants.**

$$1 : 1.2489484 :: 125 : BC = 156.11855.$$

$$1 : 1.5964824 :: 125 : AC = 199.5603.$$

Solution by Logarithmic Sines, Tangents, and Secants.

$$\text{Log. } BC = \text{log. tan. } A + \text{log. } AB - \text{log. } R.$$

$$\text{Log. tan. } 51^\circ 19' \quad - \quad - \quad - \quad 10.0965445$$

$$\text{Log. } AB \ 125 \quad - \quad - \quad - \quad 2.0969100$$

$$\text{Log. } BC \ 156.1186 \quad - \quad - \quad - \quad 2.1934545$$

In this operation $\log. R$, which is always 10.0000000, is subtracted, though not expressed. In other operations $\log. R$ will be seldom expressed, but will be added or subtracted tacitly.

$$\text{Log. } AC = \text{log. } R + \text{log. } AB - \text{log. cos. } A \text{ or } - \text{log. sine } C.$$

$$\text{Log. } R + \text{log. } AB \ 125 \quad - \quad - \quad - \quad 12.0969100$$

$$\text{Log. cos. } 51^\circ 19' \text{ or log. sine } 38^\circ 41' \quad 9.7958909$$

$$\text{Log. } AC \ 199.5603 \quad - \quad - \quad - \quad 2.3010191$$

$$\text{Log. } AC = \text{log. sec. } A + \text{log. } AB - \text{log. } R.$$

$$\text{Log. sec. } 51^\circ 19' \quad - \quad - \quad - \quad 10.2031641$$

$$\text{Log. } AB \ 125 \quad - \quad - \quad - \quad 2.0969100$$

$$\text{Log. } AC \ 199.5603 \quad - \quad - \quad - \quad 2.3000741$$

Instrumentally.

In the first proportion, extend the compasses from 45° to $51\frac{1}{4}^\circ$ on the line of tangents. That extent will

* Some tables do not contain secants, but all contain sines, cosines, and tangents.

reach from 125 to 156 on the line of numbers, and will be the length of BC.

In the second proportion, extend the compasses from $38\frac{2}{3}^{\circ}$ to 90° on the sines. That extent will reach from 125 to 199.5 on the line of numbers, and will be the length of AC.

In the third analogy, extend the compasses from 0 to $51\frac{1}{3}^{\circ}$ on the secants. That extent will reach from 125 to 199.5 on the numbers, and will be the length of AC.

Ex. 2. Given the base AB 50, and the angle at the base A $25^{\circ} 17'$, to find the rest.

Answer. Angle C = $64^{\circ} 43'$, BC = 23.617, AC = 55.297.

Case 3. Given the hypotenuse AC 415, and the base AB 249, to find the rest.

By Construction.

Draw AB = 249. At the point B erect a perpendicular. From the centre A, with the radius AC = 415, describe an arc intersecting the perpendicular in C. Then BC measured on the line of numbers is 332, and the angles A and C measured on the scale of chords, or with a protractor, are $53^{\circ} 8'$ and $36^{\circ} 53'$ nearly.

By Calculation.

$$AC : AB :: R : \text{sine } C \text{ or } \cos. A.$$

$$415 : 249 :: R : \text{sine } C \text{ or } \cos. A.$$

$$R : \tan. A :: AB : BC.$$

$$R : \tan. A :: 249 : BC.$$

Hence, $\text{sine } C = \cos. A = R \times AB \div AC$, and $BC = AB \times \tan. A \div R$.

$$415 : 249 :: 1 : 0.6 = \text{sine } C \ 36^{\circ} 52' 52''.$$

$$\text{Log. sine } C = \log. R + \log. AB - \log. AC.$$

Log. 249 + log. R	-	-	-	12.3961993
Log. 415	-	-	-	2.6180481

Log. sine C 36° 52' 52"	-	-	-	9.7781512
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Hence the angle A is 53° 7' 48".

$$1 : 1.333 :: 249 : BC = 332.$$

$$\text{Log. BC} = \text{log. tan. A} + \text{log. AB} - \text{log. R.}$$

Log. tan. 53° 7' 48"	-	-	-	10.1249384
Log. 249	-	-	-	2.3961993

Log. BC 332	-	-	-	2.5211377
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Or BC may be found as follows.

Because $AC^2 = AB^2 + BC^2$ (47. 1), $BC^2 = AC^2 - AB^2 = (AC + AB) \times (AC - AB)$ (Cor. 5. 2), therefore BC

$$= \sqrt{(AC + AB) \times (AC - AB)} =$$

$$\sqrt{(415 + 249) \times (415 - 249)} = \sqrt{664 \times 166} =$$

$$\sqrt{110224} = 332. \text{ Or, } \frac{1}{2} \log. 664 + \frac{1}{2} \log. 166 =$$

1.411084 + 1.110054 = 2.521138, the number answering to which is 332.

Ex. 2. Given the hypotenuse 50, and the base 20, to find the rest.

Answer. Angle A = 66° 25', C = 23° 35', and BC = 45.825.

Case 4. Given the base AB 53, and the perpendicular BC 67, to find the other parts of the triangle.

$$AB : BC :: R : \tan. A.$$

$$53 : 67 :: R : \tan. A.$$

$$\sin A : R :: BC : AC.$$

$$\sin A : R :: 67 : AC.$$

Hence $\tan. A = R \times BC \div AB$, and $AC = R \times BC \div \sin A$.

$$53 : 67 :: 1 : 1.26415094 = \tan. A \ 51^\circ 39' 15".$$

$$\text{Log. tan. A} = \text{log. BC} + \text{log. R} - \text{log. AB.}$$

$$\begin{array}{rcl} \text{Log. } 67 + \text{log. } R & - & 11.8260748 \\ \text{Log. } 53 & - & 1.7242759 \\ \hline \end{array}$$

$$\text{Log. tan. } A \ 51^\circ 39' 15'' \quad - \quad 10.1017989$$

$$\text{Log. } AC = \text{log. } R + \text{log. } BC - \text{log. sine } A.$$

$$\text{Log. } R + \text{log. } 67 \quad - \quad 11.8260748$$

$$\text{Log. sine } 51^\circ 39' 15'' \quad - \quad 9.8944713$$

$$\text{Log. } AC = 85.42834 \quad - \quad 1.9316035$$

$$\text{Or, since } AC^2 = AB^2 + BC^2 \ (47.1), \ AC = \sqrt{7298} = 85.428 +.$$

Also, AC may be found more easily by the secant.

$$R : \sec. \angle A :: AB : AC.$$

When a proportion can be formed, in which radius is the first term, the operation becomes more easy. Thus, the operation by the last analogy, where radius is the first term, will be more easy than that by the former analogy, where the sine of A is the first term.

Ex. 2. Given the base $AB = 20$, and the perpendicular $BC = 30$, to find the angles and the hypotenuse.

Answer. Angle $A = 56^\circ 18'$, $C = 33^\circ 42'$, $AC = 36.06$.

86. Remark. In some instances the tangents and secants are more adapted to produce accurate results than the sines and cosines. The reason of the deficiency in the sines and cosines is their small variation in certain parts of their arcs. Thus, if an arc near 90° be found in terms of its sine, and if a very small arc, or an arc near 180° , be found in terms of its cosine; the variations of the sines and cosines of these arcs are so small, that the sines and cosines will not change in the tables for many seconds of the arcs. For example, if the log. sine, or log. cosine of an arc should happen to be 9.99999998, this log. in the tables, is the sine of an arc from $89^\circ 56' 19''$ to $89^\circ 57' 8''$, or the cosine of an arc from $2' 52''$ to $3' 41''$. Consequently it is impossible to know what arc or angle, between these limits, is to be taken. The reason of this uncertainty is, that the log. sines and log. cosines, in the tables, are not continued to more than seven places of decimals, and in some tables only to five or six decimals.

In these cases it will be expedient to employ the log. tangents, or log. cotangents, which are not liable to this defect, as the difference for 1" is 42 at an arc of 45°, and greater in every other part of the quadrant.

But when a sine or cosine of this kind enters into a calculation, as one of the data, it is rather favourable than otherwise to the accuracy of the result, because any small error in the quantity of the given arc or angle will not affect the tabular value of its sine or cosine.

Again, if a required arc or angle be either very small, or very near 180°, we may express it by a sine instead of a cosine, as follows.

Because $AF = \frac{2 AL^2}{CA}$ (46. no. 3), $CF = CA - AF = CA - \frac{2 AL^2}{CA}$, that is, $\cos. ACB = r - \frac{2 \sin^2 \frac{1}{2} ACB}{r}$, therefore $r - \cos. ACB = \frac{2 \sin^2 \frac{1}{2} ACB}{r}$. Let A denote any arc or angle, then $r - \cos. A = \frac{2 \sin^2 \frac{1}{2} A}{r}$; hence $\sin^2 \frac{1}{2} A = \frac{r^2 - r \times \cos. A}{2}$, therefore $\sin \frac{1}{2} A = \frac{\sqrt{(r^2 - r \times \cos. A)}}{2}$.

Now as the log. sine of a small arc increases fast, we shall thus obtain the arc or angle A true to a second.

In like manner, if an arc very near 90° be expressed by its sine, it can be known only within certain limits, by reason of the small variation of the sine. We must therefore express the arc in terms of the cosine, which may be done by means of the formula, $\sin^2 \frac{1}{2} A = \frac{r^2 - r \times \cos. A}{2}$.

70. *Solution of the Cases of Oblique-angled Triangles.*

Fig. page 18.

Case 1. Given the angle $A 49^\circ 25'$, the angle $C 63^\circ 48'$, and the side $AB 275$, to find the rest.

The angle $B = 180^\circ - (49^\circ 25' + 63^\circ 48') = 66^\circ 47'$.

Draw $AB = 275$ from a scale of equal parts. By a scale of chords make the angle $A = 49^\circ 25'$, and the angle $B = 66^\circ 47'$. Draw AC and BC meeting each other

in C. Then AC and BC measured by the scale of equal parts will be 281·6 and 232·7.

$$\text{Sine } C : \text{sine } A :: AB : BC.$$

$$\text{Sine } 63^\circ 48' : \text{sine } 49^\circ 25' :: 275 : BC.$$

$$\text{Log. sine } A + \text{log. } AB - \text{log. sine } C = \text{log. } BC.$$

$$\text{Log. sine } 49^\circ 25' + \text{log. } 275 \quad - \quad 12\cdot3198379$$

$$\text{Log. sine } 63^\circ 48' \quad - \quad - \quad 9\cdot9529175$$

$$\text{Log. } BC = 232\cdot7665 \quad - \quad - \quad 2\cdot3669204$$

$$\text{Sine } C : \text{sine } B :: AB : AC.$$

$$\text{Sine } 63^\circ 48' : \text{sine } 66^\circ 47' :: 275 : AC.$$

$$\text{Log. sine } 66^\circ 47' + \text{log. } 275 \quad - \quad 12\cdot4026580$$

$$\text{Log. sine } 63^\circ 48' \quad - \quad - \quad 9\cdot9529175$$

$$\text{Log. } AC = 281\cdot67 \quad - \quad - \quad 2\cdot4497405$$

Ex. 2. Given the angle A $46^\circ 22'$, the angle B $79^\circ 23'$, and the side AC 135 feet, to find the rest.

Answer. The angle C = $54^\circ 15'$, the side AB = 111·47 feet, and BC = 99·41.

Case 2. Given the side AB 532, BC 358, and the angle C $107^\circ 40'$, to find the rest.

Draw BC = 358 from a scale of equal parts. By a scale of chords make the angle C = $107^\circ 40'$. From the centre B, with the radius BA = 532, intersect CA in A. Then AC measured by the scale of equal parts is 299·6, and the angles A and B measured by the scale of chords are $39^\circ 53'$ and $32^\circ 27'$.

As the angle C is obtuse, the radius BA cannot intersect CA in more points than one; therefore there cannot be two triangles; consequently this example is not ambiguous (75).

$$AB : BC :: \text{sine } C : \text{sine } A.$$

$$532 : 358 :: \text{sine } 107^\circ 40' : \text{sine } A.$$

$$\begin{array}{rcl} \text{Log. } 358 + \log. \sin 107^\circ 40' & \} & 12.5329022 \\ \text{or sup. } 72^\circ 20' & \} & \end{array}$$

$$\text{Log. } 532 \quad - \quad - \quad - \quad - \quad 2.7259116$$

$$\text{Log. } \sin A \ 39^\circ 53' \quad - \quad - \quad - \quad - \quad 9.8069906$$

As the angle C is obtuse, each of the angles A and B must be acute, or less than 90° ; therefore this example is not ambiguous (75). Hence the angle B = $180^\circ - (39^\circ 53' + 107^\circ 40') = 32^\circ 27'$.

$$\sin C : \sin B :: AB : AC.$$

$$\sin 107^\circ 40' : \sin 32^\circ 27' :: 532 : AC.$$

$$\text{Log. } \sin 32^\circ 27' + \log. 532 \quad - \quad 12.4555601$$

$$\text{Log. } \sin 107^\circ 40' \text{ or } 72^\circ 20' \quad - \quad 9.9790192$$

$$\text{Log. } AC = 299.6 \quad - \quad - \quad - \quad 2.4765409$$

Ex. 2. Given the side $AC = 20$, $BC = 30$, and the angle $A = 56^\circ 14'$, to find the rest.

Answer. The angle $B = 33^\circ 39'$, $C = 90^\circ 7'$, and the side $AB = 36.088$.

Ex. 3. Given $AC = 236$, $BC = 350$, and the angle $B = 38^\circ 40'$, to find the rest.

Draw $BC = 350$, and make the angle $B = 38^\circ 40'$. From the centre C, with the radius $CA = 236$, describe an arc, which will cut BA in two points. From the two points of intersection of BA and the radius CA draw lines to C. Then there will be two triangles, each of which will give the parts required. Hence this example is ambiguous (75). The angle C is $29\frac{1}{4}^\circ$ or $73\frac{1}{2}^\circ$, the angle A is $67\frac{3}{4}^\circ$ or $112\frac{1}{4}^\circ$, the side AB is 184 or 362.

Case 3. Given the side $AB = 176$, $BC = 133$, and the included angle $B = 73^\circ$, to find the rest.

$$\begin{array}{rcll} AB + BC : AB - BC :: \tan. \frac{1}{2} (A + C) : \tan. \frac{1}{2} (C - A). \\ 309 \quad : \quad 43 \quad :: \quad 53^\circ 30' : \tan. \frac{1}{2} (C - A). \end{array}$$

$$\begin{array}{rcl} \text{Log. tan. } 53^\circ 30' + \text{log. } 43 & - & 11.7642596 \\ \text{Log. } 309 & - & 2.4899585 \end{array}$$

$$\text{Log. tan. } \frac{1}{2} (C - A) = 10^\circ 39' \quad 9.2743011$$

Hence $\angle C = \frac{1}{2} (A + C) + \frac{1}{2} (C - A) = 53^\circ 30' + 10^\circ 39' = 64^\circ 9'$, and $\angle A = \frac{1}{2} (A + C) - \frac{1}{2} (C - A) = 53^\circ 30' - 10^\circ 39' = 42^\circ 51'$. Consequently

$$\text{Sine } C : \text{sine } B :: AB : AC.$$

$$\text{Sine } 64^\circ 9' : \text{sine } 73^\circ :: 176 : AC.$$

$$\text{Log. sine } 73^\circ + \text{log. } 176 \quad - \quad 12.2261090$$

$$\text{Log. sine } 64^\circ 9' \quad - \quad 9.9542156$$

$$\text{Log. } AC = 187.022 \quad - \quad 2.2718934$$

Ex. 2. Given one side 40, another 70, and the included angle $76^\circ 14'$, to find the rest.

Answer. The other two angles are $71^\circ 3'$ and $32^\circ 43'$, and the third side is 71.88.

Case 4. Given the three sides, AB 350, AC 240, BC 200, to find the angles. See fig. page 16.

$$AB : AC + BC :: AC - BC : AD - DB = AG.$$

$$350 : 440 :: 40 : 50.29.$$

$$AD = \frac{1}{2} (AB + AG) = 175 + 25.145 = 200.145.$$

$$DB = \frac{1}{2} (AB - AG) = 175 - 25.145 = 149.855.$$

$$AC : AD :: R : \text{sine } ACD.$$

$$240 : 200.145 :: R : \text{sine } ACD = 56^\circ 30'.$$

$$BC : BD :: R : \text{sine } BCD.$$

$$200 : 149.855 :: R : \text{sine } BCD = 48^\circ 32'.$$

Hence the angle $A = 33^\circ 30'$, $B = 41^\circ 28'$, and $ACB = 105^\circ 2'$.

Ex. 2. Given the three sides, AB = 60, AC = 50, BC = 40, to find the angles.

Answer. The angle $B = 55^\circ 46'$, $A = 41^\circ 24'$, $ACB = 82^\circ 50'$.

88. *Examples to the Cases of Plane Triangles.*

1. In a right-angled triangle given the hypotenuse 185, and one of the acute angles $32^{\circ} 40'$, to find the other parts of the triangle.
2. Given one of the sides 64, and the opposite angle $23^{\circ} 15'$.
3. Given the hypotenuse 324, and one of the sides 265.
4. Given the two sides 78 and 59.
5. In an oblique-angled triangle given one of the angles $69^{\circ} 14'$, another $46^{\circ} 27'$, and the side between them 248, to find the other parts of the triangle.
6. Given one of the angles 28° , another 114° , and the side between them 57.
7. Given one of the sides 215, another 169, and the angle opposite to the former 74° .
8. Given one of the sides 329, another 248, and the angle opposite to the latter 26° .
9. Given one of the sides 79, another 67, and the included angle $85^{\circ} 16'$.
10. Given one of the sides 241, another 175, and the included angle 103° .
11. Given one of the sides 126, another 148, and the third 96.
12. Given the three sides 119, 196, 175.

89. *Mensuration of Heights and Distances.*

The mensuration of heights and distances depends upon the use of certain instruments for taking angles, and the rules of plane trigonometry already delivered. By the joint application of these, the heights and distances of terrestrial objects may be determined, and the distances and magnitudes of the heavenly bodies may be discovered.

Horizontal and vertical angles are taken with a theodolite furnished with one or two telescopes and a vertical arc. If the circles of the theodolite be about $3\frac{1}{4}$ inches radius, the observed angles may be read off to half a minute of a degree.

If the angles to be taken be oblique to the horizon, they must be taken with a sextant, or Hadley's quadrant, which must be held in such a position that its plane may pass through both objects and the eye of the observer. Elevations are taken with the quadrant by reflecting the object from an artificial horizon.

Angles of elevation, or of depression, are commonly taken with a quadrant having its arch divided into degrees and mi-

notes, and a plummet suspended from the centre, and two sights fixed perpendicularly upon one of its radii.

Note. The use of instruments in taking angles cannot be easily taught by written directions, and is best learned from some person who is acquainted with their use.

Practical Examples.

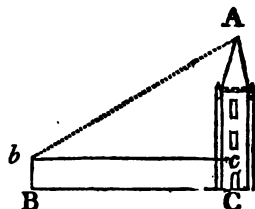
1. Having measured a distance of 200 feet from the bottom of a steeple, in a direct horizontal line, I then took the angle of elevation of its top, $\angle abc = 47^\circ 30'$. Required the height of the steeple.

$$R : bc \ 200 :: \tan. b \ 47^\circ 30' : Ac.$$

$$\text{Log. } 200 \quad - \quad - \quad 2.3010300$$

$$\text{Log. } \tan. 47^\circ 30' \quad 10.0379475$$

$$\text{Log. } Ac = 218.26 \quad 2.3389775$$



To Ac add the height of the instrument cC , then $Ac + cC = AC$ the height of the steeple.

2. The angle of elevation of a hill was found to be 46° , and 100 yards farther off, on a level with the bottom, it was 31° . Required its height.

$$\angle ABC \ 46^\circ - \angle ADC \ 31^\circ = \angle DAB \ 15^\circ.$$

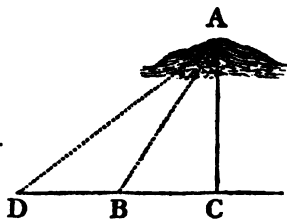
$$\text{Sine } \angle DAB \ 15^\circ : \text{sine } \angle ADC \ 31^\circ :: DB \ 100 : AB.$$

$$\text{Log. } 100 \quad - \quad - \quad 2.0000000$$

$$\text{Log. } \text{sine } 31^\circ \quad - \quad - \quad 9.7118393$$

$$\text{Log. } \text{sine } 15^\circ \quad - \quad - \quad 9.4129962$$

$$\text{Log. } AB \quad - \quad - \quad 2.2988431$$



$$R : \text{sine } \angle B \ 46^\circ :: AB : AC.$$

$$\text{Log. } AB \quad - \quad - \quad - \quad 2.2988431$$

$$\text{Log. } \text{sine } 46^\circ \quad - \quad - \quad - \quad 9.8569341$$

$$\text{Log. } AC = 143.14 \quad - \quad - \quad 2.1557772$$

3. The angles of elevation of a cloud, or other object, taken by two observers at the same time, on the same side of it, in the same vertical plane, and 880 yards distant from each other, were 64° and 35° . Required its height, and its distance from the two places of observation.

Answer. Height 935·757 yards, greater distance 1631·442, less distance 1041·125.

4. From the top of a tower 120 feet high, which was in a line with two trees on the same horizontal plain as its bottom, I took the angles formed by the perpendicular wall and a line conceived to be drawn from my eye to the bottom of each tree, and found them to be 33° and $64^\circ 30'$. Required the distance between the trees.

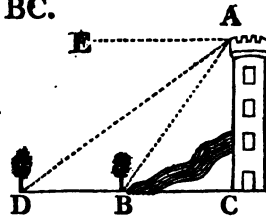
Note. An angle taken from the top of an elevated object, usually called the angle of depression, is the angle formed by a straight line conceived to be drawn from the eye to the object and another line parallel to the horizontal plane. Thus, the angle EAB is the angle of depression of the object B, which, by the nature of parallel lines, is equal to the angle ABC (29. 1). The angle BAC is the complement of EAB or ABC, and is the angle used in the calculation.

$$R : \tan. BAC \ 33^\circ :: AC \ 120 : BC.$$

$$\text{Log. } 120 \quad - \quad - \quad - \quad 2\cdot0791812$$

$$\text{Log. tan. } 33^\circ \quad - \quad - \quad - \quad 9\cdot8125174$$

$$\text{Log. } BC = 77\cdot929 \quad 1\cdot8916986$$



$$R : \tan. DAC \ 64^\circ 30' :: AC \ 120 : DC.$$

$$\text{Log. } 120 \quad - \quad - \quad - \quad 2\cdot0791812$$

$$\text{Log. tan. } 64^\circ 30' \quad - \quad - \quad - \quad 10\cdot3215039$$

$$\text{Log. } DC = 251\cdot585 \quad - \quad - \quad 2\cdot4006851$$

$$\text{Hence } DC - BC = DB = 173\cdot656.$$

5. I took the angle of elevation of a high inaccessible object, which was 58° , and at the distance of 100 yards farther, in a straight line, I again found the angle of elevation 32° . Required the height of the object, and its distance from the first station, the height of the instrument being five feet. See fig. prob. 4.

The height of the object above the ground is found to be 104.17 yards, as in prob. 2.

R or sine C 90° : cos. B 58° :: AB : BC.

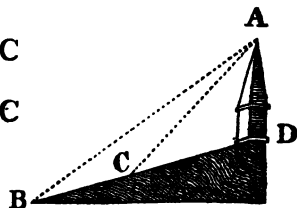
Log. cos. 58°	-	-	-	9.7242097
Log. AB (found)	-	-	-	2.0823677

Log. BC = 84.05 yards	-	-	-	1.8065774
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6. Being desirous to know the height of an obelisk erected on the top of a declivity, I measured from its bottom a distance of 40 feet, and there found the angle of elevation formed by the plain and an imaginary line drawn to the top of the object to be 41° . I then measured 60 feet farther, in the same direction, and found the angle of elevation, formed as before, $23^\circ 45'$. Required the height of the obelisk.

$$\text{Angle BAC} = \text{ACD} - \text{ABC} \\ = 41^\circ - 23^\circ 45' = 17^\circ 15'.$$

$$\text{Sine BAC } 17^\circ 15' : \text{s. ABC } 23^\circ 45' :: \text{BC } 60 : \text{AC}.$$



Log. sine $17^\circ 15'$	-	-	9.4720856
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	Complement	0.5279144
Log. sine $23^\circ 45'$	-	9.6050320
Log. 60	-	1.7781513

Log. AC = 81.488	-	-	1.9110977
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$$\text{CA} + \text{CD } 121.488 : \text{CA} - \text{CD } 41.488 :: \\ \tan. \frac{1}{2} (A + D) 69^\circ 30' : \tan. \frac{1}{2} (D - A).$$

Log. 121.488	-	-	-	2.0845333
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			Compl.	-	7.9154667
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Log. 41.488	-	-	-	1.6179225
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Log. tan. 69° 30'	-	-	-	10.4272623
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Log. tan. 42° 24' 24"	-	-	-	9.9606515
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69° 30' — 42° 24' 24" = 27° 5' 36" = angle CAD.

Sine CAD 27° 5' 36" : sine C 41° :: CD 40 : AD.

Log. sine 27° 5' 36"	-	-	-	9.6582842
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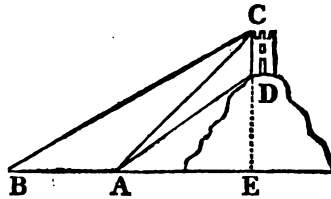
			Compl.	-	0.3417158
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Log. sine 41°	-	-	-	9.8169429
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Log. 40	-	-	-	1.6020600
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Log. AD = 57.623 feet	-	-	-	1.7607187
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7. An engineer wanting to know the height of a fort or castle erected on the summit of an inaccessible hill, took the elevation DAE 40° of the top of the fort. Then measuring a horizontal base AB of 100 feet in a direct line from the hill, he found the elevation CBE 33° 45' of the top of the castle. Required the height of the castle.



CAE — B = ACB = 17° 15'.

Sine ACB 17° 15' : s. B 33° 45' :: AB 100 : AC.

Log. sine 17° 15'	-	-	-	9.4720856
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Log. sine 33° 45'	-	-	-	9.7447390
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Log. 100	-	-	-	2.
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Log. AC	-	-	-	2.2726534
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CAE — DAE = CAD = 11°,
and 180° — ADE = ADC = 130°.

Sine ADC 130° : s. CAD 11° :: AC : DC.

Log. sine 130° or sup. 50° - 9.8842540

Comp. 0.1157460

Log. sine 11° - - - 9.2805988

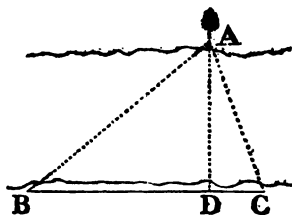
Log. AC - - - 2.2726534

Log. DC = 46.67 - - - 1.6689982

8. At the top of a castle 54 feet high, erected on a hill near the sea shore, the angle of depression = CAE of a ship at anchor was $4^\circ 52'$; and at the bottom of the castle the depression = DAE was $4^\circ 2'$. Required the horizontal distance of the vessel, and the height of the top of the castle above the sea. (See the last fig.)

Answer. Distance AE = 3690 feet, and height EC = 314 feet.

9. Desirous to know the breadth of a river I measured 100 yards in a straight line by the side of it, and at the ends of this distance I took the angles formed by it and a tree on the opposite bank, which were 53° and $79^\circ 12'$. Required the breadth of the river.



Angle BAC = $180^\circ - B - C = 180^\circ - 132^\circ 12' = 47^\circ 48'$.

Sine BAC $47^\circ 48'$: sine C $79^\circ 12'$:: BC 100 : AB.

Log. 100 - - - - 2.

Log. sine $79^\circ 12'$ - - - 9.9922385

Log. sine $47^\circ 48'$ - - - 9.8697037

Log. AB - - - - 2.1225348

R or sine D 90° : sine B 53° :: AB : AD.

Log. AB - - - - 2.1225348

Log. sine 53° - - - 9.9023486

Log. AD 105.89 - - - 2.0248834

10. Desirous to know the distance of a house, which stood on the opposite bank of a broad river, I measured 200 yards in a straight line by the side of the river, and took the angles formed at each end by the line and the house, which were $73^{\circ} 15'$ and $68^{\circ} 2'$. Required the distance of the house from each station. (See the last fig.)

Answer. The distances are found as in the first analogy of the last problem, and are $AB = 306.19$ yards, and $AC = 296.54$.

11. To find the distance between two places A, B, which are not accessible in a straight course, I seek a third place C, from which A and B can be seen; I then measure $AC = 735$ yards, and $BC = 840$ yards, and take the angle subtended by the two places $ACB = 55^{\circ} 40'$.

$$BC + AC \ 1575 : BC - AC \ 105 :: \tan. \frac{A + B}{2} \ 62^{\circ} 10'$$

$$: \tan. \frac{A - B}{2}.$$

$$\text{Log. } 105 \quad - \quad - \quad - \quad - \quad 2.0211893$$

$$\text{Log. } \tan. \ 62^{\circ} 10' \quad - \quad - \quad - \quad 10.2773793$$

$$\text{Log. } 1575 \quad - \quad - \quad - \quad 3.1972806$$

$$\text{Log. } \tan. \frac{A - B}{2} \ 7^{\circ} 12' \quad - \quad - \quad 9.1012880$$

Hence the angle $A = 69^{\circ} 22'$.

$$\text{Sine } A \ 69^{\circ} 22' : \text{sine } C \ 55^{\circ} 40' :: BC \ 840 : AB.$$

$$\text{Log. } \text{sine } 55^{\circ} 40' \quad - \quad - \quad - \quad 9.9168593$$

$$\text{Log. } 840 \quad - \quad - \quad - \quad 2.9242793$$

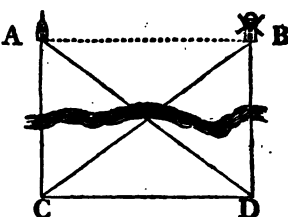
$$\text{Log. } \text{sine } 69^{\circ} 22' \quad - \quad - \quad - \quad 9.9711982$$

$$\text{Log. } AB \ 741.2 \text{ yards} \quad - \quad - \quad 2.8699404$$

12. A man being desirous to know the distance between two inaccessible objects A, B, measured a base CD of 300 yards; at C he found the angle BCD = $58^{\circ} 20'$, and $ACB = 37^{\circ}$; at D he found the angle ADC = $53^{\circ} 30'$, and $ADB = 45^{\circ} 15'$. Required the distance AB.

The angle $CAD = 180^\circ -$
 $(ACB + BCD + ADC) = 180^\circ$
 $- 148^\circ 50' = 31^\circ 10'.$

The angle $CBD = 180^\circ -$
 $(ADC + ADB + BCD) = 180^\circ$
 $- 157^\circ 05' = 22^\circ 55'.$



The angle $CDB = ADC + ADB = 98^\circ 45'.$

Sine $CAD 31^\circ 10' : \text{sine } ADC 53^\circ 30' :: CD 300 : AC.$

Log. sine $53^\circ 30'$	-	-	9.9051787
Log. 300	-	-	2.4771213
Log. sine $31^\circ 10'$	-	-	9.7139349

Log. $AC = 465.9776$ - - 2.6683651

Sine $CBD 22^\circ 55' : \text{sine } CDB 98^\circ 45' \text{ or its sup.}$
 $81^\circ 15' :: CD 300 : CB.$

Log. sine $81^\circ 15'$	-	-	9.9949158
Log. 300	-	-	2.4771213
Log. sine $22^\circ 55'$	-	-	9.5903869

Log. $BC = 761.4655$ - - 2.8816502

$BC + AC 1227.4431 : BC - AC 295.4879 ::$
 $\tan. \frac{1}{2} (CAB + ABC) 71^\circ 30' \text{ or } \cot. \frac{1}{2} ACB 18^\circ 30' :$
 $\tan. \frac{1}{2} (CAB - ABC).$

Log. 295.4879	-	-	2.4705397
Log. cot. $18^\circ 30'$	-	-	10.4754801
Log. 1227.4431	-	-	3.0890013

Log. $\tan. 35^\circ 44'$ - - 9.8570185

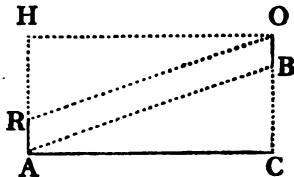
Hence $\angle ABC = 71^\circ 30' - 35^\circ 44' = 35^\circ 46'.$

Sine $ABC 35^\circ 46' : \text{sine } ACB 37^\circ :: AC 465.9776 : AB.$

Log. sine 37°	-	-	9.7794630
Log. 465.9776	-	-	2.6683651
Log. sine $35^\circ 46'$	-	-	9.7667739

Log. $AB = 479.79 \text{ yards}$ - - 2.6810542

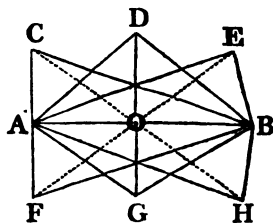
90. In certain trigonometrical operations, when a base is measured on sloping ground, it is sometimes necessary to reduce it to the corresponding horizontal line. Let AB be the measured base, OB a theodolite, and AR a staff equal to the height of OB . Let HOR be the angle of depression of the top of R below the horizontal line HO . If CO be perpendicular to HO , then AC , which is parallel to HO , will be the horizontal base corresponding to AB . Hence $R : AB :: \cos. HOR$ or $BAC : AC$.



If AB be 300 yards, and the angle of depression HOR 5° , the horizontal line AC will be 298.9 yards, which differs from the measured base AB by 1.1 yard only. Therefore when the measured base is inclined to the horizon in a small angle a reduction is not necessary, unless accuracy be required.

91. The distances of the most remarkable places in a town, or of several villages from one another, may be determined by the rules of trigonometry; also the plan of a camp, or of a country may be taken.

Let A, C, D, E, B, H, G, F , be several objects, whose situations are to be laid down in a map. Find a convenient position AB for the base, from which all the objects can be seen; and let the base be as long as possible, in proportion to the most distant object. From the extremity A of the base take the angles $EAB, DAB, CAB, HAB, GAB, FAB$; and from the other extremity B take the angles $CBA, DBA, EBA, FBA, GBA, HBA$. The common base AB and the angles of all the triangles being now known, the sides AC, AD, AE , &c., and consequently the points C, D, E , &c. may be determined by article 54.

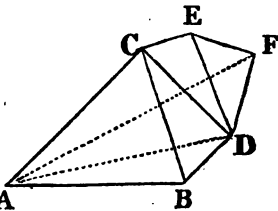


To insure the accuracy of the operation all the objects C, D, E , &c. should be intersected from some third station O , in the base AB ; otherwise the figure or plan of the survey may appear correct, when it is not so; and there will be no means of discovering whether the angles have been justly taken.

92. *To carry on a Measurement by a series of Triangles.*

A measurement may be carried on, or the distance of any two remote objects may be found, by means of a series of triangles, formed from a measured base. In this manner the trigonometrical survey of a country is generally performed.

Let AB represent the measured base, and C, D any two objects visible from the two stations A, B. If the angles CAB, CBA, DAB, DBA, be taken with a theodolite, or other instrument, we can find the sides BC, BD by article 54, and the angle BDA. The angles ADBA, CBA being known, their difference CBD is also known. From the two sides BC, BD, and the included angle CBD we can find the third side CD, and the two angles BCD, BDC.



If E, F be two other objects which are visible from the stations C, D, we can find the lengths of EF, DF in the same manner. And thus may the measurement be continued from one base to another to any distance.

To render the conclusion less inaccurate, the mensuration from one base to another may be carried on by different sets of triangles, leading to the same two objects. For instance; instead of C, D other two proper objects might have been chosen, by means of which the length of EF might have been found in the same manner. The mean result of several sets of triangles will probably approach nearer the truth than the result of a single set.

The distance AF between the first station and the last may be also determined. For the two sides AB, BD, and the included angle ABD being known, the side AD may be found; therefore the two sides AD, DF, and the included angle ADF being known, the side AF may be found.

In surveys of this kind it is expedient to observe every angle of all the triangles, if the situations will permit; because the difference between 180° and the sum of the three angles of each triangle will enable us, in some measure, to judge of the accuracy of the operations.

All the principal distances ought to be laid down from a scale of equal parts, because a triangle can be protracted less inaccurately from the sides than from the angles.

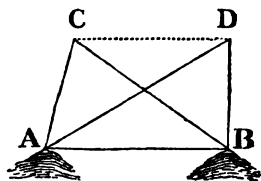
After a series of triangles has been carried on to some distance, in this manner, it is expedient to measure the interval

between two objects, which has been already determined by calculation. By this means the error of the computed distance may be discovered. This measured line is called the *base of verification*.

93. The area or content of any quantity of land measured in this manner may be found as follows.

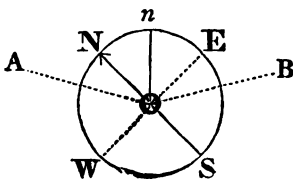
Let P denote a perpendicular drawn from the angle A on the opposite side BC, then $R = 1 : \sin A :: AB : P = AB \times \sin A$; hence, by mensuration, the area of the triangle $ABC = \frac{1}{2} AB \times BC \times \sin A$. In the same manner the area of $BCD = \frac{1}{2} BC \times BD \times \sin B$, of $CDE = \frac{1}{2} CD \times DE \times \sin C$, of $EDF = \frac{1}{2} ED \times DF \times \sin D$. The sum of the several triangles will be the area of the whole tract of land.

94. It sometimes happens, in making a survey, that, when the distance between two objects, C, D has been determined, it is required to find the distance between two eminences A, B, which are conveniently situated for extending the series of triangles. In this case, measure the angles CAD, CAB, DBC, DBA. Now, as there are not sufficient data in any of the triangles to compute the unknown parts, we must assume a value for AB, and thence compute the value of CD, as above. Then, the computed value of CD : its true value :: the assumed value of AB : its true value. For, by changing the value of AB, while the angles at A and B remain the same, the whole figure will continue similar to itself, and consequently AB will vary as CD.



95. Military sketches, or small surveys, may be made by means of a compass fixed on the top of a staff. The staff being stuck in the ground, so that the needle of the compass may play freely, the angular distances, or bearings, as they are commonly called, must be taken from the magnetic meridian. But the compass must not be used when accuracy is required.

Let NS represent the needle, or magnetic meridian, n the true north point, and E, W the east and west points; then if the sights of the compass be directed to the object A, and the angle NOA be 40° , the object is said to bear



NW 40° . If the sights be directed to the object B, and the angle NOB be 110° , the object is said to bear NE 110° .

But the angular distances, or bearings of objects, are commonly expressed by the letters which denote the points of the compass. Thus, if an object bear in the direction of the point of the compass, which is denoted by the letters NNE, it makes an angle with the meridian of $22^\circ 30'$, from the north toward the east; and if an object bear SW, it makes an angle with the meridian of 45° , from the south toward the west.

Example 1. As I was travelling in England I perceived a fort on a high hill, which bore from me NE $22\frac{1}{2}$ degrees; and having gone 20 miles in the direction NW $67\frac{1}{2}$ degrees, I saw the fort again, which now bore NE $56\frac{1}{2}$ degrees. Required the distance of the fort from each place of observation.

Let A be my first station, MAm the meridian, C the fort, AC its direction on bearing, B my second station, BC the bearing of the fort = angle m DC.

Angle ACB = m DC — DAC = $56^\circ 15' - 22^\circ 30' = 33^\circ 45'$.

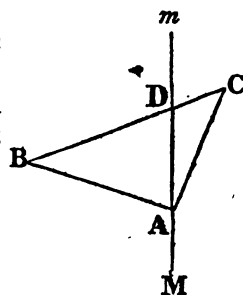
Angle BAC = CAD + DAB = $22^\circ 30' + 67^\circ 30' = 90^\circ$.

Angle ABC = $180^\circ - \text{BAC} - \text{ACB} = 180^\circ - 123^\circ 45' = 56^\circ 15'$.

Sine C $33^\circ 45'$: sine BAC $90^\circ ::$ AB 20 : BC.

Log. sine 90°	-	-	-	10.
Log. 20	-	-	-	1.301030
Log. sine $33^\circ 45'$	-	-	-	9.744739

Log. BC = 36 miles - - - 1.556291



Sine C $33^{\circ} 45'$: sine B $56^{\circ} 15'$:: AB 20 : AC.

Log. sine $56^{\circ} 15'$ - - - 9.919846

Log. 20 - - - 1.301030

Log. sine $33^{\circ} 45'$ - - - 9.744739

Log. AC = 29.93 miles - - 1.476137

2. At a certain place A St. Paul's church C, in London, bore from me NE $11\frac{1}{2}$ degrees; and when I had travelled 15 miles farther to B, in the direction NW $22\frac{1}{2}$ degrees, it bore NE $50^{\circ} 37'$. Required its distance at the last place of observation. *Answer.* 13.14 miles.

3. From a ship at sea a point of land C bore NE by N. The ship sailed NW 23 miles, and the same point then bore E by N. Required the distance of the point from the ship at both stations.

The angle BAC = NE by N + NW = $33^{\circ} 45' + 45^{\circ} = 78^{\circ} 45'$.

The angle C = m DC — DAC = E by N — NE by N = $78^{\circ} 45' - 33^{\circ} 45' = 45^{\circ}$.

Hence the angle B = $56^{\circ} 15'$.

Then (54) the distance of the land from the first station of the ship is 27.045 miles, and from the second station 31.902 miles.

4. As a ship was sailing along the coast of Carolina a cape was observed to bear NW by N, and another NNE. The ship then sailing ENE $\frac{1}{2}$ E 21 miles, the first cape bore WNW, and the second N by W $\frac{1}{2}$ W. Required the distance between the two capes.

Answer. 28.0847 miles.

96. Problems for the Exercise of Learners.

The following problems are intended as exercises for students, and therefore the answers of most of them have been omitted. The reader will be able to resolve them without much difficulty, by comparing them with similar questions, of which the solutions are given above. They are not however

deemed necessary, and the solution or the omission of them is submitted to the judgment of the teacher.

1. The angle of elevation of a tower is 65° , and the distance from the place of observation to the bottom of the tower is 80 feet. Required the height of the tower.

2. A ladder 40 feet long will reach the top of a house, when the foot of the ladder is placed 15 feet from the bottom of the house. Required the height of the house, and the inclination of the ladder to the horizon.

3. Required the length of a scaling ladder, to scale a tower whose angle of elevation is $67^\circ 41'$, across a ditch 16 feet wide.

4. A tower, whose height is 100 feet, subtends an angle of 25° . Required the distance of the observer from the bottom of the tower.

5. What angle will the top of the same tower (100 feet high) subtend, when the observer is twice as far from it? and what angle will it subtend when he is only half as far from it?

6. What is the altitude of the sun when a man's shadow is half his height? and what is the altitude of the sun when a man's shadow is double his height?

$1 : 2 :: R : \tan. \text{sun's altitude. Answer. } 63^\circ 26'.$

$2 : 1 :: R : \tan. \text{sun's altitude. Answer. } 26^\circ 34'.$

7. How far will the shadow of an upright pillar extend at noon, upon a level pavement, the meridian altitude of the sun being 50° , and the height of the pillar 100 inches?

Answer. 83 inches.

8. Standing on the top of a tower $136\frac{1}{2}$ feet high, I observed a tree at a distance on the plain, and found its angle of depression $22^\circ 40'$. Required the distance of the tree from the bottom of the tower.

9. From the top of a castle on the shore, 156 yards above the level of the sea, the angle of depression of a ship is $56^\circ 40'$. Required the distance of the ship both from the bottom and the top of the castle.

10. There are three towns A, B, C; at A the towns B and C make an angle of $20^\circ 12'$; at B the towns A and C make an angle of $38^\circ 30'$; and the distance between A and B is $2\frac{1}{2}$ miles. Required the distance between C and A, and C and B.

11. As I was walking on the sea-shore I observed an island, which made an angle of $95^\circ 20'$ with a straight line along the shore. At the distance of 650 links farther the island made an angle of $75^\circ 40'$ with my first station. Required its distance from the first station.

12. A man travels from A to B, 3 miles and $\frac{7}{10}$; then, bending a little to the right, he goes from B to C, 4 miles and $\frac{7}{10}$; at C he observes that A and B make an angle of $29^\circ 16'$. Required his least distance from home. *Answer.* 7 miles.

13. A man travels from A to B, 3 miles and $\frac{7}{10}$; returning in a mist he loses his way, and going a little too much on the right hand comes to C, which is 4 miles and $\frac{7}{10}$ from B. The weather now cleared up, and he could see both A and B, and observed that they made an angle of $22^\circ 16'$. Required his distance from home. *Answer.* 1.2 mile.

14. There are three towns A, B, C. The distance between A and C is 3 miles and 3 furlongs, and the distance between B and C is 4 miles and 5 furlongs. Between A and B lies a large wood, which prevents these towns from being seen from each other, and their distance from being measured. However A and B are visible from C, and there make an angle of $71^\circ 2'$. What is the distance between A and B? and how can a vista be cut through the wood, so that A and B may be seen from each other?

15. There are three towns A, B, C. The distance of A and B is 5 miles, of B and C 9 miles, of C and A 7 miles. What are their respective bearings from each other?

Answer. Angle A = $95^\circ 44\frac{1}{3}$, B = $50^\circ 42\frac{1}{3}$, C = $33^\circ 33\frac{2}{3}$.

16. How shall I plant three trees, so that the angles which every two trees make with each other may be 50, 60, and 70 degrees?

The distances between the trees will be to one another respectively as the sines of the angles which are opposite to them (56), and therefore will vary indefinitely in this proportion. Now the natural sines of 50° , 60° , and 70° , to radius 1, are .76604, .86603, .93969; therefore these numbers, or any other in the same proportion, will represent the proportional, not the absolute distances between the trees. Suppose x to be the distance between two trees opposite to the angle 50° , then $s. 50^\circ : s. 60^\circ :: x : x \times s. 60^\circ \div s. 50^\circ = \text{distance of two trees opposite to } 60^\circ$. Let $x = 20$ yards, then $\log. 20 + \log. s. 60^\circ - \log. s. 50^\circ = 1.354307$, the number answering to which is 22.61 yards, the side or distance opposite to the angle 60° .

17. How must three trees A, B, C be planted in a triangular form, so that the angle at A may be double of the angle at B, and the angle at B double of the angle at C; and a line of 100 yards may go round them?

It is evident that the angles are to one another as the num-

bers 1, 2, 4. Consequently there are given the sum of the three angles 180° (32. 1), and their ratio, to find the angles.

Let $x = \angle C$, then $2x = \angle B$, and $4x = \angle A$.

Hence $7x = 180^\circ$, therefore $x = \frac{180^\circ}{7}$, therefore

$$2x = \frac{2 \times 180^\circ}{7}, \text{ and } 4x = \frac{4 \times 180^\circ}{7}.$$

Now the sides of a plane triangle are respectively as the sines of their opposite angles. Hence there are given the sum of three numbers 100, and their ratio, to find the numbers.

Let $a = \frac{180^\circ}{7}$, and y = least side, or distance between two trees, then

$$s. a : s. 2a :: y : \frac{y \times s. 2a}{s. a} = \text{middle side,}$$

$$\text{and } s. a : s. 4a :: y : \frac{y \times s. 4a}{s. a} = \text{longest side.}$$

$$\text{Hence } y + \frac{y \times s. 2a}{s. a} + \frac{y \times s. 4a}{s. a} = 100,$$

$$\text{therefore } y \times s. a + y \times s. 2a + y \times s. 4a = 100 \times s. a,$$

$$\text{therefore } y = \frac{100 \times s. a}{s. a + s. 2a + s. 4a}.$$

This formula is not adapted to calculation by logarithms; therefore *natural* sines must be applied, as follows.

$$s. a = s. \frac{180^\circ}{7} = s. 25^\circ 42' 51'' \quad - \quad - \quad - \quad - \quad = 43388$$

$$s. 2a = s. \frac{2 \times 180^\circ}{7} = s. 51^\circ 25' 43'' \quad - \quad - \quad - \quad - \quad = 78183$$

$$s. 4a = s. \frac{4 \times 180^\circ}{7} = s. 102^\circ 51' 26'' = s. \text{sup. } 77^\circ 8' 34'' = 97493$$

$$\text{Sum} = 219064$$

$$\text{Hence } y = \frac{100 \times s. a}{s. a + s. 2a + s. 4a} = \frac{43388}{219064} = 19.8 \text{ yards.}$$

$$\text{Consequently } \frac{y \times s. 2a}{s. a} = \frac{19.8 \times 78183}{43388} = 35.677 \text{ yards,}$$

$$\text{and } \frac{y \times s. 4a}{s. a} = \frac{19.8 \times 97493}{43388} = 44.488 \text{ yards.}$$

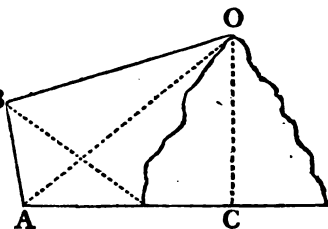
18. Let A and B be two inaccessible objects, whose distance is required. At two stations C and D, from which the objects and also the other station can be seen, the angles ACB $62^\circ 12'$, BCD $41^\circ 8'$, and ADB $60^\circ 49'$, ADC $34^\circ 51'$, are taken

with an instrument; and the distance CD between the stations is found to be 562 yards. *Answer.* 729·7 yards.

Note. This problem is useful in taking surveys of coasts or harbours.

19. The angle of elevation of a certain tower was observed to be 20° . The observer intended to measure his distance from the bottom of the tower, to determine its height; but when he had measured 85 feet in a direct line toward the tower, he was stopped by a ditch; therefore he again took the altitude of the tower, and found it to be $31^\circ 34'$. Required the height of the tower, and the distance of the observer from the bottom, when he was stopped by the ditch.

20. Wanting to know the height of a hill OC, and its distance AC from the station A, I measured a base AB of B 298 yards, on ground nearly level, and at the extremities A, B observed the angles contained between the summit O and each station, BAO $42^\circ 17'$,



and ABO $79^\circ 29'$; also at A I took the elevation of the hill OAC $4^\circ 51'$. Required the distance AC, and the height CO. *Answer.* AC 344·6 yards, and CO 29·2.

21. There is a certain mountain, the height of which above the neighbouring plain is required. To determine its height two stations were taken upon the plain in sight of the highest cliff; and in sight of each other. At the first station A the angle of elevation of the cliff was observed to be $22^\circ 49'$, and the angle between the cliff and the second station B was found to be $39^\circ 10'$. At the second station B the angle between the cliff and the first station A was observed to be $48^\circ 45'$. Lastly, the distance between the two stations was found to be 240 yards. Required the height of the cliff, and its distance from the nearer station; also that point in the base line (produced if necessary) which is nearest to the cliff.

It is evident that the station B is nearer the cliff than A, because the angle at B is greater than the angle at A, and therefore the side opposite to B is greater than the side opposite to A. Moreover the point in the base produced, where the angle between the cliff and the first station is a right angle, will be nearest to the cliff.

22. If A and C be two stations on sloping ground, 410 yards distant, O an object on the top of a hill, the angle OCA

$= 79^\circ 29'$, the angle $OAC = 63^\circ 11'$, the angle of elevation at $A = 6^\circ 36'$, at $C = 5^\circ 22'$; what are the horizontal distances of the object from the two stations, and its height above the level of each station?

Sine $O : s. A :: AC : CO$, and

$s. O : s. C :: AC :: AO$.

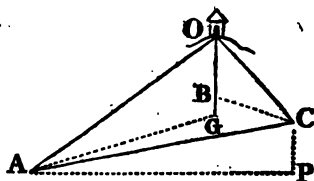
$R : s. \angle \text{elevation at } A :: AO :$

$OG = 76.4 \text{ yards.}$

$R : \cos. \angle \text{elevation at } A :: AO$

$: AG = 660.3 \text{ yards.}$

In like manner are found OB
 $= 56.43 \text{ yards, and } CB = 600.73 \text{ yards.}$



23. At a mile-stone N, on the ascending road NS, a person observed the angle SNW between the next mile-stone S and the windmill W, on the top of a hill, and found it to be $46^\circ 37'$; he also took the angle of elevation of W $3^\circ 49'$. At the next mile-stone S he took the angle NSW $91^\circ 4'$ between the first mile-stone and the mill. Required the horizontal distance NP, and the height PW of the object.

Answer. $NP = 2608 \text{ yards, and } PW = 174.$

24. To complete some chorographical observations about Cambridge, in England, it was necessary to measure exactly the distance between the steeple of St. Mary's church and the observatory at Trinity college; but that distance could not be measured in a direct line by reason of the houses between these two objects. Therefore two stations were chosen in the fields behind Trinity college, from each of which the observatory, St. Mary's, and the other station could be all seen at once. At the first station the angle made by the top of the observatory and the top of St. Mary's steeple was $14^\circ 34'$, the angle between the top of the observatory and the second station was $60^\circ 50'$, and the angle between the top of St. Mary's steeple and the second station was $46^\circ 16'$. At the second station the angle between the top of the observatory and the first station was $96^\circ 44'$, the angle between the top of St. Mary's and the first station was $115^\circ 23'$. Lastly, the base line or distance between the two stations was exactly measured, and found to be 908.36 feet.

What is the horizontal distance between the observatory and St. Mary's steeple from these data?

1. *Answer.* If the two buildings were of the same height the distance would be 674.62 feet.

2. If the observatory be 73·50 feet high, and St. Mary's steeple 97·25 feet high, the distance is 674·20 feet, which differs from the other distance by less than half a foot.

This problem is similar to prob. 12, page 47, and may be resolved in the same manner.

SECTION III.*

I. *Of the Signs of Trigonometrical Lines.*

97. The several changes in the algebraic signs of all the lines described in and about a circle must be particularly observed in the application of trigonometry to the solution of astronomical and physical problems. We shall therefore trace all the changes of the signs of those lines particularly, as follows. See plate, figure 1.

Let AHDL be a circle having the sines, tangents, &c. represented as in the figure. Suppose one extremity A of an arc AB to remain fixed, while the other extremity B passes successively over the circumference of the circle, from A through the points H, D, L, to A again.

The sine begins at A, and increases from 0 (nothing) during the first quadrant AH, till it becomes equal to radius at the point H. Then it decreases during the second quadrant HD, till it again becomes 0 at the end of it. After this the sine passes to the other side of the diameter AD; and therefore, being reckoned affirmative before, is now to be considered negative. During the third quadrant DL the sine *Bf* increases till it becomes equal to radius, but is negative, or equal to —radius. During the fourth quadrant LA it decreases from —radius till it becomes 0, when the arc is 360 degrees. After this the sine is affirmative, and changes continually as at first, in every revolution of the point B round the circumference of the circle.

It appears from an inspection of the figure, that the sine *BF* increases faster in the first part AB of the quadrant AH, than when it approaches near the end of it, at H; and, on the contrary, that it decreases slower in the first part H*b* of the second quadrant HD, than when it arrives near the end of it, at

* The learner may omit this section at the first reading of plane trigonometry, but must afterward read the whole with attention, if he intends to prosecute the study of mathematics, because it is necessary to be known in many parts of mechanical philosophy.

D. This variation of the rate of the increase or decrease of the sine arises from the convexity of the circle in certain positions of the sine.

98. The cosine is equal to radius, when the arc is 0, and continually decreases during the first quadrant AH, till it becomes 0 at the end of it, at H. The cosine being computed from the centre of the circle, will be negative after it passes the centre; therefore the cosine Cf, which lies in an opposite direction to the cosine CF, will be negative. This negative cosine increases during the second quadrant HD, at the end of which it is equal to — radius. It decreases negatively during the third quadrant DL, at the end of which it is 0. In the fourth quadrant LA it becomes positive, and increases till it is equal to radius, as before.

Since the cosine of the arc AB is equal to the sine of its complement BH, it follows, reversedly, from what has been said respecting the variation of the rate of increase of the sines, that the cosine CF decreases slower in the first part AB of the quadrant AH, than when it approaches near the end of it, at H; and, on the contrary, that it increases faster in the first part Hb of the second quadrant HD, than when it arrives near the end of it, at D.

99. The tangent AT becomes negative as often as it meets the radius CB produced on the side of the point A or diameter AD opposite to that in which it is first drawn; and as this happens both when the arc AB becomes greater than AH, and also when, by its further increase, it is greater than the arc AHDL, it follows that the tangents of all arcs in the first and third quadrants AH, DL, are positive, and that the tangents of all arcs in the second and fourth quadrants HD, LA, are negative.

The tangent at the beginning of the arc is 0, and increases to *infinity* during the first quadrant AH; that is, no line can be assigned, how great soever its length, but an angle under 90 degrees may be found, whose tangent shall exceed that line; consequently the tangent has no limit to its increase, as the sine has. In the second quadrant HD the tangent At is negative, for the tangents being computed from the point A, the tangent At will be in a direction opposite to AT. During this quadrant the tangent decreases from an infinite negative to 0. In the third quadrant DL it is again affirmative, and it creases from 0 to infinity, as in the first quadrant. In the fourth quadrant LA it decreases from an infinite negative to 0, as the second quadrant; after which it becomes affirmative, and then increases as in the first quadrant.

From the nature of the circle it is apparent that the tangent AT increases more slowly about the middle of the quadrant AH than in any other part of it, and that it increases fast as the point B approaches near H , having no limit to its increase, like the sine, but admitting all possible degrees of magnitude from 0 to infinity.

100. The secant becoming negative as often as the revolving radius CB passes the centre C , changes its sign like the cosine, and is affirmative in the first and fourth quadrants, and negative in the second and third quadrants.

In the first quadrant AH the secant increases from radius to infinity. In the second quadrant HD it is negative; for the secant has its origin at the centre C , and its length is computed from the centre to its concurrence with the tangent. In the first quadrant the length of the secant is computed from C toward B and T ; but in the second quadrant it is reckoned, on the revolving radius, from C toward E and t , in a contrary direction, and therefore is negative. During the second quadrant this negative secant decreases from infinity till it becomes equal to radius. In the third quadrant DL it increases from radius to infinity, but continues negative; for the intersection which this revolving radius (produced) makes with the tangent continues on the same side (on that revolving line), with respect to the centre C , both for the second and third quadrants. In the fourth quadrant LA this intersection changes to the opposite part of the revolving radius (produced), namely, the same as at first; therefore in the fourth quadrant the secant is affirmative, and decreases from infinity till it becomes equal to radius, as it was at the beginning of the first quadrant.

Thus the secant has the same algebraic signs as the cosine, in the same quadrants.

101. We may observe here, that the sine and cosine never exceed radius, that the secant and cosecant are never less than radius, and that the tangent admits all degrees of magnitude. Moreover, all these lines change their directions as often as they become either infinite or nothing. When they become infinite, their increase is at its utmost limit; after which they change their direction, and decrease. When they become 0, their decrease is at its utmost limit, and then they increase again in a contrary direction. Thus, these quantities change their algebraic signs, whenever they pass through a state of infinity, or a state of nothingness.

30. As the cotangent HK is computed from the point H , in the same manner as the tangent AT is computed from A , the

former will evidently vary in its direction and length like the latter, being affirmative in the first and third quadrants, AH, DL, and negative in the second and fourth quadrants, HD, LA. In the first quadrant AH the cotangent HK decreases from infinity to 0, and in the second quadrant HD it increases negatively from 0 to infinity. It becomes affirmative in the third quadrant DL, and decreases from infinity to 0. In the fourth quadrant LA it increases negatively from 0 to infinity, as in the second quadrant.

102. The cosecant CK changes its sign in the same manner as the sine, being affirmative in the first and second quadrants AH, HD, and negative in the third and fourth quadrants DL, LA. In the first quadrant AH it decreases from infinity to radius, and in the second quadrant HD it increases from radius to infinity. In the third quadrant DL it decreases negatively from infinity to radius, and in the fourth quadrant LA it again increases negatively from radius to infinity.

103. The versed sine AF increases from 0 during the first and second quadrants AH, HD, till it becomes the diameter AD, which is its utmost limit. It then decreases during the third and fourth quadrants DL, LA, till it becomes 0. Being always computed in the same direction, from A toward D, it is always affirmative.

104. The changes of the algebraic signs of the several trigonometrical lines are exhibited in the following table.

	1st quad.	2d quad.	3d quad.	4th quad.
Sine and cosecant	+	+	—	—
Cosine and secant	+	—	—	+
Tangent and cotangent	+	—	+	—

105. The values of these lines at the end of each quadrant of the circle may be exhibited in a table, as follows.

	0°	90°	180°	270°	360°
Sine	0	r	0	$-r$	0
Cosine	r	0	$-r$	0	r
Tangent	0	∞	0	∞	0
Cotangent	∞	0	∞	0	∞
Secant	r	∞	$-r$	∞	r
Cosecant	∞	r	∞	$-r$	∞

106. In many analytical investigations it is common to employ, indifferently, arcs of all magnitudes, whether positive or negative, greater or less than 360° ; in which cases their sines,

cosines, &c., may be derived from fig. 1 in the same manner nearly as those of the simple arcs.

Thus, if to any arc AB there be added one or more circumferences of the circle, it is manifest that they will terminate again exactly in the point B, and that the arc AB, so augmented, will have the same positive or negative sine, cosine, &c. as the single arc AB. Whence, if C denote the whole circumference of a circle, or 360° , and x any arc AB; then $\sin x = \sin (C + x) = \sin (2C + x) = \sin (3C + x) = \&c.$ The case is the same in respect of the cosine, tangent, &c.

107. Also, if two arcs AB, AE, be taken in opposite directions on the circumference of the circle AHDL, one arc being considered positive, the other negative; their sines, in this case, will be equal, but will be affected with contrary algebraic signs; and the cosines of both arcs will be the same, namely, CF. Hence, if any negative arc AE be denoted by $-a$, then

$$\begin{array}{lcl} \text{Sine } (-a) = -\sin a & \left| & \text{Cos. } (-a) = +\cos a \\ \text{Tan. } (-a) = -\tan a & \left| & \text{Cot. } (-a) = -\cot a \\ \text{Sec. } (-a) = +\sec a & \left| & \text{Cosec. } (-a) = -\text{cosec. } a \end{array}$$

If $360^\circ - a$ be substituted for $-a$, in any expression of this kind, it will be necessary only to consider those arcs which are positive.

108. By the inspection of fig. 1 it appears that the sine, tangent, and secant of any arc AB, are of the same magnitudes as the cosine, cotangent, and cosecant of its complement BH, and *vice versa*; therefore the values of the former of these lines may be expressed in terms of the latter, as follows. $\sin a = \cos. (90^\circ - a)$, $\cos. a = \sin (90^\circ - a)$, &c.

Moreover, the sine, cosine, &c. of any arc are of the same magnitudes as the sine, cosine, &c. of its supplement ($360^\circ - a$); therefore these lines may be expressed, in a similar manner, with their proper signs, as follows. $\sin a = \sin (180^\circ - a)$, $\cos. a = -\cos. (180^\circ - a)$, &c.

109. By substituting $90^\circ - a$ for a in these last forms, it will appear that the sine, cosine, &c. of any arc or angle below 90° , is equal to the sine, cosine, &c. of an arc or angle as much above 90° as the former is under 90° . Thus, $\sin (90^\circ - a) = \sin (90^\circ + a)$, $\cos. (90^\circ - a) = -\cos. (90^\circ + a)$, &c.

In each of these forms we must attend to the change of the signs, when the arc or angle a is greater than 90° . Since the $\cos. a = \sin (90^\circ - a)$, it is plain that, if we know how to value the sine in all possible cases, we shall be able to value the cosine, and thence all the rest of the trigonometrical lines.

110. The changes of the signs of the trigonometrical lines may be shown in a neat and concise manner, as follows. Vince's Trigonometry.

Suppose the arc AB to begin at A , and in the quadrant AH assume the sine, cosine, tangent, secant, cotangent, and cosecant, all positive. If a line, supposed to be positive in one direction, vanish, and then be set off in a contrary direction, it becomes negative. Hence CF , the cosine of the arc AB , is positive; but Cf , the cosine of the arc Ab or $A\beta$, is negative.

In the arc AHD the sine is set off in the same direction, and therefore is positive; but at D it vanishes, and in the arc DLA is set off in an opposite direction; therefore it now becomes negative.

The tangent = radius $\times$ sine $\div$ cosine (41). Now the radius is always positive, and from A to H the sine, cosine, tangent, &c. are positive by supposition. At H the cosine is 0, and the tangent and secant vanish, for they become parallel, and therefore never meet. Hence a quadrant, or an arc of 90 degrees, has neither a tangent nor a secant (according to the definition). From H to D the sine is positive, and the cosine is negative; therefore the tangent is negative. From D to L the sine and cosine are both negative, therefore the tangent is positive. From L to A the sine is negative, and the cosine is positive; therefore the tangent is negative.

The secant = radius² $\div$ cosine; therefore the secant has always the same sign as the cosine.

The cotangent = radius² $\div$ tangent; therefore the cotangent has always the same sign as the tangent.

The cosecant = radius² $\div$ sine; therefore the cosecant has always the same sign as the sine.

The versed sine is always set off from A in the same direction, and therefore continues positive through the whole circle.

Hence the changes of the algebraic signs of the several trigonometrical lines may be conveniently exhibited in a table, as in art. 104.

111. If the arc AB be considered as positive, the arc AE , set off from A in a contrary direction, will be negative. The arcs AB , AE are equal, and have the same cosine CF , and the same versed sine AF ; but the sine EF being set off in a direction contrary to that of the sine BF , becomes negative; that is, the sine of a negative arc AE has a sign contrary to that of the sine of a positive arc AB of the same value.

The tangent = radius $\times$ sine $\div$ cosine, and the cotangent = radius $\times$ cosine $\div$ sine, and the cosecant = radius $\div$ sine. Hence it appears from what has been said respecting the sine of a negative arc, that the tangent, cotangent, and cosecant will have the sign for a negative arc different from the sign which they have for a positive arc.

The secant = radius $\div$ cosine; therefore the secant has the same sign both for a positive arc and a negative arc; for it has been shown that the sign of the cosine is the same for both arcs.

Hence, if A, B denote two arcs, of which A is the less, the signs of the sine, tangent, cotangent, and cosecant of A — B will be contrary to those exhibited in the preceding table.

112. Any arc or angle and its supplement have the same sine (34); therefore, in trigonometrical computations, when the quantity required is found in terms of the sine, the case is ambiguous, unless the ambiguity be removed by some other consideration. But if the quantity sought be expressed by a cosine, tangent, or cotangent, there is no ambiguity, because a positive cosine, tangent, or cotangent denotes an arc or angle less than 90° ; but a negative cosine, &c. denotes an arc or angle between 90° and 180° (for in trigonometry every arc or angle must be less than 180°). In all trigonometrical calculations it is necessary to attend to the signs of the quantities.

II. *Trigonometrical Theorems.**

PROP.

113. If there be three such arcs (plate, fig. 3), AB, AC, AD, that BC, the difference between the first and second arcs, is equal to CD, the difference between the second and third; then the radius is to the cosine of the common difference, BC, as the sine of the middle arc, AC, is to half the sum of the sines of the extreme arcs, AB, AD.

Draw CE to the centre E. Draw BF, CG, DH perp. to the radius AE, and they will be the sines of the arcs AB, AC,

* This subject is treated with perspicuity and elegance by Legendre, *Elements de Geometrie*, p. 338.

AD. Draw BD cutting CE in I. Draw IK perp. to AE, and IM, BL perp. to DH.

Because the arc BD is bisected in C, EC will be perp. to the chord BD, and therefore will bisect it in I (3. 3); therefore BI will be the sine, and EI the cosine of BC or CD. Since BD is bisected in I, and IM is parallel to BL, LD is also bisected in M (2. 6). Now $BF = HL$, therefore $BF + DH = DH + HL = DL + 2 LH = 2 LM + 2 LH = 2 MH = 2 IK$; therefore $IK = \frac{1}{2}(BF + DH)$ = half the sum of the sines of the extreme arcs AB, AD.

By similar triangles CGE, IKE, $CE : EI :: CG : IK$, that is, $R : \cos. BC :: s. AC : \frac{1}{2}(s. AB + s. AD)$.

114. Cor. 1. Hence $R \times (s. AB + s. AD) = 2 s. AC \times \cos. BC$, that is, the rectangle under radius and the sum of the sines of two arcs is equal to twice the rectangle under the sine of half the sum and the cosine of half the difference of those arcs.

115. Cor. 2. If the point B coincide with A, then $R : \cos. BC :: s. BC : \frac{1}{2} s. BD$, that is, radius is to the cosine of any arc, as the sine of the arc is to half the sine of twice the arc.

Let A denote any arc or angle, then $\sin A \times \cos. A = \frac{1}{2} R \times \sin 2 A$. If $R = 1$, then $\frac{1}{2} s. 2 A = s. A \times \cos. A$, or $s. 2 A = 2 s. A \times \cos. A$.

116. The preceding theorem is one of four which, when arithmetically expressed as follows, are frequently used in the application of trigonometry to the solution of the more difficult problems in the higher parts of mathematics.

If in the figure the arc $AC = A$, $BC = B$, and radius $EC = 1$, then $AD = A + B$, and $AB = A - B$; therefore by the proposition, $1 : \cos. B :: s. A : \frac{1}{2} s. (A + B) + \frac{1}{2} s. (A - B)$, therefore $s. A \times \cos. B = \frac{1}{2} s. (A + B) + \frac{1}{2} s. (A - B)$.

117. Because BF, IK, DH are parallel, the straight lines BD, FH are cut proportionally (2. 6); therefore FH, the difference of the straight lines FE, HE is bisected in K; therefore, as was shown in the proposition, KE is half the sum of FE and HE, that is, of the cosines of the arcs AB and AD. But, by the similar triangles EGC, EKI, $EC : EI :: EG : EK$, that is, $R : \cos. BC :: \cos. AC : \frac{1}{2} \cos. AD + \frac{1}{2} \cos. AB$, or, $1 : \cos. B :: \cos. A : \frac{1}{2} \cos. (A + B) + \frac{1}{2} \cos. (A - B)$, that is,

the radius : cos. of the common difference :: cos. of the middle arc : half the sum of the cosines of the extreme arcs.

Hence $\cos. A \times \cos. B = \frac{1}{2} \cos. (A + B) + \frac{1}{2} \cos. (A - B)$.

118. Again, the triangles IDM, CEG, having the angles DMI, EGC right, and the angles DIM, EIK or ECG, equal, because the angle EIM is the complement of each, are equiangular; therefore $EC : CG :: DI : IM$. Now IM is half the difference between the cosines FE and EH; therefore

$R : s. AC :: s. BC : \frac{1}{2} \cos. AB - \frac{1}{2} \cos. AD$,
or, $1 : s. A :: s. B : \frac{1}{2} \cos. (A - B) - \frac{1}{2} \cos. (A + B)$,
therefore $s. A \times s. B = \frac{1}{2} \cos. (A - B) - \frac{1}{2} \cos. (A + B)$.

119. In the same triangles ECG, DIM, $EC : EG :: ID : DM$. Now DM is half the difference of the sines DH and BF; therefore $R : \cos. AC :: s. BC : \frac{1}{2} s. AD - \frac{1}{2} s. AB$,
or, $1 : \cos. A :: s. B : \frac{1}{2} s. (A + B) - \frac{1}{2} s. (A - B)$,
therefore $\cos. A \times s. B = \frac{1}{2} s. (A + B) - \frac{1}{2} s. (A - B)$.

120. Hence if A, B denote any two arcs, and the radius = 1, then

- I. $S. A \times \cos. B = \frac{1}{2} s. (A + B) + \frac{1}{2} s. (A - B)$.
- II. $\cos. A \times \cos. B = \frac{1}{2} \cos. (A - B) + \frac{1}{2} \cos. (A + B)$.
- III. $S. A \times s. B = \frac{1}{2} \cos. (A - B) - \frac{1}{2} \cos. (A + B)$.
- IV. $\cos. A \times s. B = \frac{1}{2} s. (A + B) - \frac{1}{2} s. (A - B)$.

121. Cor. If $A = 30^\circ$, then $s. A = \frac{1}{2} R$ (45); therefore theorem I becomes $\cos. B = s. (30^\circ + B) + s. (30^\circ - B)$, therefore $s. (30^\circ + B) = \cos. B - s. (30^\circ - B)$.

If $A = 60^\circ$, then $\cos. A = \frac{1}{2} R$ (45); therefore theorem IV becomes $s. B = s. (60^\circ + B) - s. (60^\circ - B)$, therefore $s. (60^\circ + B) = s. (60^\circ - B) + s. B$.

122. From these four theorems other four are deduced.

By adding the first and fourth theorems,

$$s. A \times \cos. B + \cos. A \times s. B = s. (A + B).$$

By subtracting the fourth theorem from the first,

$$s. A \times \cos. B - \cos. A \times s. B = s. (A - B).$$

By adding the second and third theorems,

$$\cos. A \times \cos. B + s. A \times s. B = \cos. (A - B).$$

By subtracting the third theorem from the second,

$$\cos. A \times \cos. B - s. A \times s. B = \cos. (A + B).$$

The last four theorems are presented in one view, as follows.

- I. $S. (A + B) = s. A \times \cos. B + \cos. A \times s. B.$
 II. $S. (A - B) = s. A \times \cos. B - \cos. A \times s. B.$
 III. $\cos. (A + B) = \cos. A \times \cos. B - s. A \times s. B.$
 IV. $\cos. (A - B) = \cos. A \times \cos. B + s. A \times s. B.$

123. Since $s. A \times \cos. B = \frac{1}{2} s. (A + B) + \frac{1}{2} s. (A - B)$ (120), if $A + B = S$, and $A - B = D$, then $A = \frac{1}{2}(S + D)$ (57), and $B = \frac{1}{2}(S - D)$; therefore,
 $s. \frac{1}{2}(S + D) \times \cos. \frac{1}{2}(S - D) = \frac{1}{2} s. S + \frac{1}{2} s. D.$

Now, because S and D denote any two arcs, to preserve the former notation, they may be called A and B , which will also express any arcs whatever. Thus,

$$s. \frac{1}{2}(A + B) \times \cos. \frac{1}{2}(A - B) = \frac{1}{2} s. A + \frac{1}{2} s. B,$$

$$\text{or, } 2 s. \frac{1}{2}(A + B) \times \cos. \frac{1}{2}(A - B) = s. A + s. B.$$

In the same manner is deduced

$$2 \cos. \frac{1}{2}(A + B) \times \cos. \frac{1}{2}(A - B) = \cos. B + \cos. A, \text{ from}$$

theor. II, art. 120,

$$2 s. \frac{1}{2}(A + B) \times s. \frac{1}{2}(A - B) = \cos. B - \cos. A, \text{ from th. III,}$$

$$2 \cos. \frac{1}{2}(A + B) \times s. \frac{1}{2}(A - B) = s. A - s. B, \text{ from th. IV.}$$

In all these theorems the arc B is supposed less than A .

124. Theorems of the same kind with respect to the tangents of arcs may be deduced from the preceding theorems.

Because the tangent of any arc is equal to the sine $\div$ cosine

$$(41), \tan. (A + B) = \frac{s. (A + B)}{\cos. (A + B)}.$$

$$\text{But } s. (A + B) = s. A \times \cos. B + \cos. A \times s. B,$$

$$\text{and } \cos. (A + B) = \cos. A \times \cos. B - s. A \times s. B;$$

$$\text{therefore } \tan. (A + B) = \frac{s. A \times \cos. B + \cos. A \times s. B}{\cos. A \times \cos. B - s. A \times s. B}.$$

Divide both the numerator and denominator of this fraction by $\cos. A \times \cos. B$, and it becomes

$$\frac{\frac{s. A}{\cos. A} + \frac{s. B}{\cos. B}}{1 - \frac{s. A}{\cos. A} \times \frac{s. B}{\cos. B}} = \frac{\tan. A + \tan. B}{1 - \tan. A \times \tan. B} \text{ by substitution.}$$

$$\text{Hence } \tan. (A + B) = \frac{\tan. A + \tan. B}{1 - \tan. A \times \tan. B}.$$

$$\text{In like manner, } \tan. (A - B) = \frac{\tan. A - \tan. B}{1 + \tan. A \times \tan. B}.$$

125. By art. 123, $s. A + s. B = 2 s. \frac{1}{2}(A+B) \times \cos. \frac{1}{2}(A-B)$,
and $s. A - s. B = 2 \cos. \frac{1}{2}(A+B) \times s. \frac{1}{2}(A-B)$,

$$\text{therefore } \frac{s. A + s. B}{s. A - s. B} = \frac{s. \frac{1}{2}(A+B) \times \cos. \frac{1}{2}(A-B)}{\cos. \frac{1}{2}(A+B) \times s. \frac{1}{2}(A-B)} = \frac{\tan. \frac{1}{2}(A+B)}{\tan. \frac{1}{2}(A-B)}.$$

Hence $s. A + s. B : s. A - s. B :: \tan. \frac{1}{2}(A+B) : \tan. \frac{1}{2}(A-B)$, that is, the sum of the sines of any two arcs : difference of the sines :: tangent of half the sum of the arcs : tangent of half their difference.

If $A = 90^\circ$, and $B = 2x$, then
 $R + s. 2x : R - s. 2x :: \tan. (45^\circ + x) : \tan. (45^\circ - x).$

In the same manner, $\frac{\cos. B + \cos. A}{\cos. B - \cos. A} = \frac{\cos. \frac{1}{2}(A+B) \times \cos. \frac{1}{2}(A-B)}{s. \frac{1}{2}(A+B) \times s. \frac{1}{2}(A-B)} = \frac{\cot. \frac{1}{2}(A+B)}{\tan. \frac{1}{2}(A-B)}.$
Hence $\cos. B + \cos. A : \cos. B - \cos. A :: \cot. \frac{1}{2}(A+B) : \tan. \frac{1}{2}(A-B).$

In the same manner, $\frac{s. A + s. B}{\cos. A + \cos. B} = \frac{s. \frac{1}{2}(A+B) \times \cos. \frac{1}{2}(A-B)}{\cos. \frac{1}{2}(A+B) \times \cos. \frac{1}{2}(A-B)} = \tan. \frac{1}{2}(A+B).$

126. By art. 122 $\sin(A+B) : \sin(A-B) :: s. A \times \cos. B + \cos. A \times s. B : s. A \times \cos. B - \cos. A \times s. B ::$ (by dividing the last two terms by $\cos. A \times \cos. B$)

$$\frac{s. A}{\cos. A} + \frac{s. B}{\cos. B} : \frac{s. A}{\cos. A} - \frac{s. B}{\cos. B} :: \tan. A + \tan. B : \tan. A - \tan. B :: (41) \frac{R^2}{\cot. A} + \frac{R^2}{\cot. B} : \frac{R^2}{\cot. A} - \frac{R^2}{\cot. B} :: \cot. B + \cot. A : \cot. B - \cot. A.$$

Again, $\cos. (A+B) : \cos. (A-B) :: \cos. A \times \cos. B - s. A \times s. B : \cos. A \times \cos. B + s. A \times s. B ::$ (by dividing by $s. A \times \cos. B$) $\frac{\cos. A}{s. A} - \frac{s. B}{\cos. B} : \frac{\cos. A}{s. A} + \frac{s. B}{\cos. B} :: \cot. A - \tan. B : \cot. A + \tan. B :: \frac{R^2}{\tan. A} - \frac{R^2}{\cot. B} : \frac{R^2}{\tan. A} + \frac{R^2}{\cot. B} :: \cot. B - \tan. A : \cot. B + \tan. A.$

127. In all the preceding theorems the radius R is supposed equal to unity, because in this manner the propositions are most concisely expressed, and are also most readily applied to trigonometrical calculation. But if it be required to enunciate any of them geometrically, the multiplier R , which has disappeared by being supposed $= 1$, must be restored; and it will always be evident from inspection in what terms the multiplier R is wanted. Thus, in theorem I, art. 120, 2 s. $A \times \cos. B = s. (A + B) + s. (A - B)$ is a true proposition, taken arithmetically; but if taken geometrically it is absurd, unless we supply the radius as a multiplier of the terms on the right hand of the sign of equality. It then becomes 2 s. $A \times \cos. B = R \times s. (A + B) + R \times s. (A - B)$, or, twice the rectangle under the sine of A and the cos. of B is equal to the rectangle under radius and the sum of the sines of $A + B$ and $A - B$.

In general, the number of *linear multipliers*, that is, of lines whose numerical values are multiplied together, must be the same in every term, otherwise we shall compare unlike magnitudes with one another, which is absurd.

The propositions in this section are useful in many of the higher branches of the mathematics, and are the foundation of what is called the *Arithmetic of Sines*. Almost every branch of the mathematical sciences has been simplified and extended by the application of this calculus; but it has chiefly facilitated the investigations in physical astronomy.

III. Construction of Trigonometrical Tables.*

128. In all the calculations performed by the preceding rules, tables of sines, tangents, &c. are necessarily employed.

A trigonometrical canon is a table exhibiting the length of the sine, tangent, &c. to every degree and minute of the quadrant, from $1'$ to 90° , the radius being supposed equal to unity, and to be divided into 10000000 or more decimal parts.

In constructing the tables the first thing to be done, is to compute the sine and cosine of an arc of $1'$, or the sine and cosine of the least arc in the tables.

* This subject is treated fully and clearly in Horsley's *Mathematics*, vol. III, p. 120; also in Legendre's *Elements de Geometrie*, p. 358.

PROBLEM.

129. Given the radius of a circle equal to unity, to find the sine and cosine of $1'$. Fig. page 10.

Let AB be an arc of 60° , and let it be bisected in M , and MC drawn to the centre. Then the chord $AB = \text{radius (26)} = 1$, and $AL = \frac{1}{2} AB = \frac{1}{2} \text{ radius}$, that is, radius being unity, the sine of $30^\circ = \frac{1}{2}$. Hence the cosine of $30^\circ = CL =$

$\sqrt{CA^2 - AL^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = .8660254$. The cosine of 30° being found, the sine of 15° may be found as follows.

Let AB be the chord of an arc of 30° , then $AB^2 = 2 CM \times ML$ (art. 47. cor. 1), therefore $AB = \sqrt{2 CM \times ML}$, therefore $\frac{1}{2} AB = AL = \sqrt{\frac{1}{2} CM \times ML}$. But $ML = CM - CL = .1339746$, therefore the sine of $15^\circ = \sqrt{\frac{1}{2} CM \times ML} = \sqrt{.5 \times .1339746} = .2588190$.

The sine of 15° being thus found, the sine of $7^\circ 30'$ may be found in the same manner, and thence the sine of $3^\circ 45'$, and so on, till after 12 bisections of the arc of 60° we find the sine of an arc of $52'' 44''' 3'''' 45'''''$, which is .000255663462.

From the sine of an arc the cosine may be found, for $\sqrt{\text{radius}^2 - \text{sine}^2} = \text{cosine}$. Hence the cosine of $52'' 44''' 3'''' 45'''''$ is found.

130. The sine of a very small arc is nearly equal to the length of the arc. For the number of the sides of an equilateral polygon inscribed in a circle may be so great, that the perimeter of the polygon and the circumference of the circle will differ by a line less than any given line (cor. 6. 1. sup.), or, which is the same thing, will be to each other nearly in the ratio of equality. Therefore their like parts will also be nearly in the ratio of equality; consequently the side of the polygon will be to the arc of the circle which it subtends nearly in the ratio of equality; therefore half the side of the polygon will be to half the arc which it subtends nearly in the ratio of equality, that is, the sine of any very small arc will be to the arc itself nearly in the ratio of equality. Therefore if two arcs be very small, the first arc will be to the second very nearly as the sine of the first arc is to the sine of the second.

Hence the sine of an arc of $52'' 44''' 3'''' 45'''''$ being known, the sine of an arc of $1'$ may be found; for $52'' 44''' 3'''' 45''''' : 1' :: \text{sine of the former arc } .000255663462 : \text{sine of the lat-}$

ter. Thus, the sine of an arc of $1'$ is found to be $\cdot0002908882$. Hence the cosine of $1'$ is $\cdot999999958$.

131. The sine of $1'$ being thus found, the sine of $2'$, $3'$, or of any number of minutes, may be found by the following proposition.

EC : EG :: DI : DM :: DB : DL (119)

or R : cos. AC :: 2 s. BC : s. AD — s. AB.

Let AB = a , BC = b , then AC = $a + b$, AD = $a + 2b$.

Hence R = 1 : cos. ($a + b$) :: 2 s. b : s. ($a + 2b$) — s. a ,

therefore s. ($a + 2b$) — s. a = 2 cos. ($a + b$) $\times$ s. b ,

therefore s. ($a + 2b$) = 2 cos. ($a + b$) $\times$ s. b + s. a .

Now suppose $b = 1'$, $a = 0'$, $1'$, $2'$, $3'$, $4'$, &c.; then

$$\text{sine } 2' = 2 \cos. 1' \times \text{sine } 1',$$

$$\text{sine } 3' = 2 \cos. 2' \times \text{sine } 1' + \text{sine } 1',$$

$$\text{sine } 4' = 2 \cos. 3' \times \text{sine } 1' + \text{sine } 2'.$$

In general, $\text{sine } n' = 2 \cos. (n - 1)' \times s. 1' = s. (n - 2)'$.

Hence, the sine and cosine of $1'$ being known, we obtain the sine of $2'$, and consequently the cosine of $2'$. From the sine of $1'$ and the cosine of $2'$, we can find the sine and cosine of $3'$. Thus may a table of the sines and cosines for every minute of the quadrant be computed; and as the multiplier, sine $1'$, is constant, the operation is not tedious.

When the arcs differ by more than $1'$ the method is the same.

132. When you have found the sine and cosine for every minute of the arc to 30° , proceed as follows. By art. 121 $\text{sine } (30^\circ + B) = \cos. B - \text{sine } (30^\circ - B)$. Take $B = 1'$, $2'$, $3'$, $4'$, &c. then

$$\text{sine } 30^\circ 1' = \cos. 1' - \text{sine } 29^\circ 59',$$

$$\text{sine } 30^\circ 2' = \cos. 2' - \text{sine } 29^\circ 58',$$

$$\text{sine } 30^\circ 3' = \cos. 3' - \text{sine } 29^\circ 57',$$

$$\text{\&c.} \qquad \qquad \qquad \text{\&c.}$$

Thus find all the sines, and thence all the cosines, as far as 45° . Then all the sines and cosines to 90° will be found; for the sine and cosine of an arc above 45° are respectively equal to the cosine and sine of an arc as much below 45° , one of these arcs being the complement of the other; that is, the cosine of $44^\circ = \text{sine of } 46^\circ$, the cosine of $43^\circ = \text{sine of } 47^\circ$; also the sine of $44^\circ = \cosine \text{ of } 46^\circ$, the sine of $43^\circ = \cosine \text{ of } 47^\circ$, &c.

133. The sines and cosines of all arcs to 90° being found, the versed sines are found by subtracting the cosines from radius (32).

134. The tangent of any arc is found by dividing the sine by the cosine of the same arc (41). When the tangents have been found in this manner as far as 45° , the tangents from 45° to 90° may be computed more conveniently, by another rule.

The tangent of an arc above 45° is the cotangent of an arc as much below 45° , and radius is a mean proportional between the tangent and the cotangent of any arc (41); therefore if D denote the difference between any arc and 45° ,
 $\tan. (45^\circ - D) : 1 :: 1 : \tan. (45^\circ + D)$; therefore

$$\tan. (45^\circ + D) = \frac{1}{\tan. (45^\circ - D)}.$$

135. The tangents being found, the cotangents are known, for the tangent of an arc is the cotangent of the complement of that arc. Thus, the $\tan. 50^\circ = \cot. 40^\circ$.

136. The radius is a mean proportional between the cosine and the secant of any arc, therefore if A be any arc,

$$\sec. A = \frac{1}{\cos. A}.$$

137. The secants being found, the cosecants are known, for the secant of an arc is the cosecant of its complement. Thus, the secant of $55^\circ = \text{cosec. } 35^\circ$.

138. When the sines have been found to every minute of the quadrant, the intermediate sines to every ten seconds of a degree, or even to every second, may be easily deduced by taking proportional differences. For the differences of the sines of arcs which are nearly equal will be as the differences of the arcs. Hence, if the sines of $39^\circ 45'$ and $39^\circ 46'$ be known, the sine of $39^\circ 45' 13''$ may be found as follows. Find the difference of the two given arcs, and also the difference of their sines; then $60''$ (the difference between $39^\circ 45'$ and $39^\circ 46'$) : $13'' :: \text{dif. of the sines of } 39^\circ 45' \text{ and } 39^\circ 46' : \text{excess of the sine of } 39^\circ 45' 13'' \text{ above the sine of } 39^\circ 45'$. Add the excess thus found to the sine of $39^\circ 45'$, and you will obtain the sine of $39^\circ 45' 13''$.

This method will serve in all cases, when the arcs between which the interpolation is to be made are so great that the difference of $1'$ bears an inconsiderable proportion to either arc.

139. The natural sines, cosines, &c. being computed by the preceding rules, find their logarithms in the common tables of logarithms of the natural numbers, and add 10 to the indexes of their logarithms; the results will be the logarithms of the sines, cosines, &c.; and will be the same as those in the tables of logarithmic sines, tangents, &c. The logarithms in the

tables are supposed to be the logarithms of the sines, tangents, &c. to the radius 10000000000, which is the reason that 10 is to be added to the indexes of the logarithms of the natural numbers.

140. In the computation of tables the calculation should extend to two figures more than you intend to insert in the tables, that the last figure in the tables may have its nearest value, either greater or less than the true value. Thus, if the tables are to contain seven places of figures, you must compute nine figures; then the last two figures will always indicate the nearest figure to be inserted in the seventh place in the tables. If the computed value be 847962356, the nearest value to seven figures is 8479624. If the figure in the eighth place of the computed value be not 5, then eight figures will be sufficient; for if the eighth figure be greater or less than 5, the figure in the seventh place must be greater or less by unity than the value found in the calculation.

141. The old tables of logarithms contained the secants and versed sines, as well as the sines and tangents. But they are of so little use, and, when occasion for them occurs, are so easily derived from the other lines, that they are now omitted, and sines and tangents only are exhibited in the most improved modern tables.

Since the sine and cosine, the tangent and cotangent of any arc greater than a quadrant, are the same as the sine and cosine, the tangent and cotangent of its supplement, it is sufficient to exhibit the sines and tangents of arcs not exceeding the quadrant. If the canon give the sine or tangent of the arc AM, it also gives the sine or tangent of its supplement MBD. If I want to know the sine or tangent of $127^{\circ} 15'$, I look for the sine or tangent of $52^{\circ} 45'$.

142. As many of the theorems or formulæ, which have been demonstrated in the first and third sections, are often employed in mathematical inquiries, it will be useful to collect them into a table, with some others which are deduced from them by simple processes.

Table of the most useful Trigonometrical Formulae.

$$\text{Sine}^2 a + \text{cos.}^2 a = r^2$$

$$\text{Sine } (a \pm b) = \frac{\text{sine } a \text{ cos. } b \pm \text{cos. } a \text{ sine } b}{r}$$

$$\text{Cos. } (a \pm b) = \frac{\text{cos. } a \text{ cos. } b \mp \text{sine } a \text{ sine } b}{r}$$

$$\text{Sine } a \text{ cos. } b = \frac{1}{2} r \text{ sine } (a + b) + \frac{1}{2} r \text{ sine } (a - b)$$

$$\text{Cos. } a \text{ sine } b = \frac{1}{2} r \text{ sine } (a + b) - \frac{1}{2} r \text{ sine } (a - b)$$

$$\text{Cos. } a \text{ cos. } b = \frac{1}{2} r \text{ cos. } (a + b) + \frac{1}{2} r \text{ cos. } (a - b)$$

$$\text{Sine } a \text{ sine } b = -\frac{1}{2} r \text{ cos. } (a + b) + \frac{1}{2} r \text{ cos. } (a - b)$$

$$\text{Sine } a + \text{sine } b = \frac{2}{r} \text{ sine } \frac{1}{2} (a + b) \text{ cos. } \frac{1}{2} (a - b)$$

$$\text{Sine } a - \text{sine } b = \frac{2}{r} \text{ cos. } \frac{1}{2} (a + b) \text{ sine } \frac{1}{2} (a - b)$$

$$\text{Cos. } a + \text{cos. } b = \frac{2}{r} \text{ cos. } \frac{1}{2} (a + b) \text{ cos. } \frac{1}{2} (a - b)$$

$$\text{Cos. } a - \text{cos. } b = -\frac{2}{r} \text{ sine } \frac{1}{2} (a + b) \text{ sine } \frac{1}{2} (a - b)$$

$$\text{Sine } 2a = \frac{2 \text{ sine } a \text{ cos. } a}{r}$$

$$\text{Sine } \frac{1}{2} a = \frac{1}{2} \sqrt{2r^2 - 2r \text{ cos. } a}$$

$$\text{Cos. } 2a = \frac{\text{cos.}^2 a - \text{sine}^2 a}{r} = \frac{2 \text{ cos.}^2 a - r^2}{r}$$

$$\text{Cos. } \frac{1}{2} a = \frac{1}{2} \sqrt{2r^2 + 2r \text{ cos. } a}$$

$$\text{Sine}^2 a = \frac{1}{2} r (r - \text{cos. } 2a)$$

$$\text{Cos.}^2 a = \frac{1}{2} r (r + \text{cos. } 2a)$$

$$\text{Sine}^2 a - \text{sine}^2 b = \text{cos.}^2 b - \text{cos.}^2 a =$$

$$\text{sine } (a + b) \text{ sine } (a - b)$$

$$\text{Cos.}^2 a - \text{cos.}^2 b = \text{cos. } (a + b) \text{ cos. } (a - b)$$

$$\text{Tan. } a = \frac{r \text{ sine } a}{\text{cos. } a}$$

$$\text{Cot. } a = \frac{r^2}{\text{tan. } a} = \frac{r \text{ cos. } a}{\text{sine } a}$$

$$\text{Sec. } a = \frac{r^2}{\text{cos. } a}$$

$$\text{Cosec. } a = \frac{r^2}{\text{sine } a}$$

$$\text{Tan. } (a \pm b) = \frac{r \text{ sine } (a \pm b)}{\text{cos. } (a \pm b)} = \frac{r^2 (\text{tan. } a \pm \text{tan. } b)}{r^2 \mp \text{tan. } a \text{ tan. } b}$$

$$\begin{aligned}\text{Tan. } a + \text{tan. } b &= \frac{r^2 \sin(a+b)}{\cos. a \cos. b} \\ \text{Tan. } a - \text{tan. } b &= \frac{r^2 \sin(a-b)}{\cos. a \cos. b} \\ \text{Cot. } a + \text{cot. } b &= \frac{r^2 \sin(a+b)}{\sin a \sin b} \\ \text{Cot. } a - \text{cot. } b &= -\frac{r^2 \sin(a-b)}{\sin a \sin b} \\ \text{Tan.}^2 a - \text{tan.}^2 b &= \frac{r^4 \sin(a+b) \sin(a-b)}{\cos.^2 a \cos.^2 b} \\ \text{Cot.}^2 a - \text{cot.}^2 b &= -\frac{r^4 \sin(a+b) \sin(a-b)}{\sin.^2 a \sin.^2 b}\end{aligned}$$

$$\begin{aligned}\frac{\sin a + \sin b}{\sin a - \sin b} &= \frac{\tan. \frac{1}{2}(a+b)}{\tan. \frac{1}{2}(a-b)} \\ \frac{\sin a + \sin b}{\cos. a + \cos. b} &= \frac{\tan. \frac{1}{2}(a+b)}{r} \\ \frac{\sin a + \sin b}{\cos. a - \cos. b} &= -\frac{\cot. \frac{1}{2}(a-b)}{r} \\ \frac{\sin a - \sin b}{\cos. a + \cos. b} &= \frac{\tan. \frac{1}{2}(a-b)}{r} \\ \frac{\sin a - \sin b}{\cos. a - \cos. b} &= -\frac{\cot. \frac{1}{2}(a+b)}{r} \\ \frac{\cos. a + \cos. b}{\cos. a - \cos. b} &= -\frac{\cot. \frac{1}{2}(a-b)}{\tan. \frac{1}{2}(a+b)} = -\frac{\sec. a + \sec. b}{\sec. a - \sec. b} \\ \frac{\sin a}{r + \cos. a} &= \frac{\tan. \frac{1}{2} a}{r} & \frac{\sin a}{r - \cos. a} &= \frac{\cot. \frac{1}{2} a}{r} \\ \sin a &= \frac{r \tan. a}{\sqrt{r^2 + \tan.^2 a}} & \cos. a &= \frac{r^2}{\sqrt{r^2 + \tan.^2 a}}\end{aligned}$$

$$R = \sin 1^g = \cos. 0^g = \tan. \frac{1}{2}^g = \cot. \frac{1}{2}^g = \sec. 0^g = \text{cosec. } 1^g$$

$$\sin a = \frac{1}{2} \text{ cord } 2a$$

$$\text{Cord } a = 2 \sin \frac{1}{2} a$$

$$\sin(1^g + a) = + \cos. a$$

$$\cos.(2^g + a) = + \sin a$$

$$\sin(2^g + a) = + \sin a$$

$$\cos.(2^g + a) = - \cos. a$$

$$\sin(3^g + a) = - \cos. a$$

$$\cos.(3^g + a) = + \sin a$$

$$\sin(4^g + a) = + \sin a$$

$$\cos.(4^g + a) = + \cos. a$$

END OF PLANE TRIGONOMETRY.

SPHERICAL TRIGONOMETRY.

INTRODUCTION.

THE following definitions and properties of spherical triangles belong to spherical geometry, and are premised here as principles on which the demonstrations of the propositions in spherical trigonometry depend.*

Definitions.

1. A sphere or globe is a solid contained under one uniform round surface, which is every where equally distant from a point within it called the centre.

A sphere may be conceived to be formed by the revolution of a semicircle about its diameter, which remains unmoved, and is called the axis of the sphere.

2. A diameter of a sphere is a straight line passing through the centre, and terminated both ways by the convex surface.

3. The circles of the sphere are of two denominations, *great* circles, and *small* circles.

A great circle of the sphere is that which divides its surface into two equal parts.

A small circle of the sphere is that which divides its surface into two unequal parts.

Thus, the equator of a common globe is a great circle, and any parallel of latitude is a small circle.

4. Hence the plane of any great circle passes through the

* These definitions and elementary propositions, with their demonstrations, will be annexed to the second edition of Playfair's Geometry, under the title of Elements of Spherical Geometry. They constitute no part of spherical trigonometry, according to the genuine meaning of the terms which express the title of that science; and therefore cannot, with propriety, be prefixed to a regular and systematic treatise of spherical trigonometry.

centre of the sphere, and divides the sphere into two equal parts.

5. The poles of any circle are the two extremities of that diameter of the sphere which is perpendicular to the plane of the circle.

6. Hence either pole of any circle is equidistant from every part of its circumference; and each pole of a great circle is 90° from the circumference.

7. A spherical angle is an angle on the surface of a sphere, contained between the arcs of two great circles which intersect each other.

8. The measure of a spherical angle is the arc of a great circle intercepted between the two arcs which form the angle, and drawn at the distance of 90° from the angular point.

9. A spherical triangle is a portion of the surface of a sphere contained by the arcs of three great circles which intersect one another.

Note. The three arcs are called the sides of the triangle, and the three angles which every two arcs form by their intersection are called its angles.

Also, both the sides and the angles of spherical triangles are computed in degrees, minutes, and seconds, in the same manner as the angles of plane triangles.

10. A right-angled spherical triangle is that which has one right angle, or an angle of 90° .

11. A quadrantal spherical triangle is that which has one of its sides a quadrant, or 90° .

12. An oblique-angled spherical triangle is that which has each of its sides, or angles, greater or less than 90° .

13. Any two sides, or angles, of a spherical triangle are said to be *like*, or *of the same kind*, or *of the same affection*, when they are both greater or less than 90° .

14. If one side or angle of a spherical triangle be equal to, or greater than 90° , and the other side or angle less, they are said to be *unlike*, or *of different kinds*, or *of different affections*.

Properties of the Sphere.

15. Every section of a sphere by a plane passing through it is a circle.

16. The centre of a sphere is the centre of all its great circles, and its axis is the common section of all the great circles which pass through its two extremities.

17. A great circle can be drawn through any two points on

the surface of a sphere, and a small circle can be drawn through any three points on its surface.

18. All parallel circles of the sphere have the same pole; and no two great circles can have a common pole.

19. Any two great circles of the sphere cut each other twice at the distance of 180° , and make the angles at the intersections equal.

20. A great circle of the sphere is perpendicular to any other circle, when its plane is perpendicular to the plane of the other; and conversely.

21. A great circle passing through the poles of any other great circle cuts the other circle at right angles; and if a great circle cut any other circle at right angles it will pass through its poles.

Note. Most of these principles will be evident by inspecting the nature and position of the circles drawn on an artificial globe. The sixth article becomes obvious by observing that all the meridians pass through the north and south poles of the globe, and are perpendicular to the equator, and to all the parallels of latitude.

22. If two arcs of great circles intersect each other, the vertical or opposite angles will be equal.

23. An angle made by the intersection of any two great circles of the sphere is equal to the angle of inclination of the planes of those circles.

24. The distance of the poles of any two great circles of the sphere is equal to the angle of inclination of the planes of those circles.

General Properties of Spherical Triangles.

25. Any side, or any angle, of a spherical triangle is less than 180° , or two quadrants.

26. The greater side is opposite to the greater angle, and the less side to the less angle.

27. The sum of any two sides is greater than the third side, and their difference is less than the third side.

28. The difference of any two sides is less than 180° , or a semicircle; and the sum of the three sides is less than 360° , or two semicircles.

29. The sum of the three angles is greater than 180° , or two right angles; and less than 540° , or six right angles.

30. The sum of any two angles is greater than the supplement of the third angle.

31. A spherical triangle is equilateral, isosceles, or scalene, according as its angles are all equal, or only two of them equal, or all unequal.

32. If each of the three angles be acute, or right, or obtuse, then each of the three sides will be less than 90° , or equal to 90° , or greater than 90° ; and conversely.

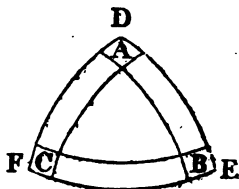
33. Half the sum of any two sides is of the same kind as half the sum of their opposite angles.

Or, the sum of any two sides is of the same kind, in respect of 180° , as the sum of their opposite angles.

34. If three arcs of great circles be described from the angular points of any spherical triangle, as poles, the sides and angles of the new triangle, so formed, will be the supplements of the opposite angles and sides of the former triangle; and conversely.

Thus, $DE = 180^\circ - C$, $EF = 180^\circ - A$, $FD = 180^\circ - B$, and $D = 180^\circ - BC$, $E = 180^\circ - AC$, $F = 180^\circ - AB$.

Again, $AB = 180^\circ - F$, $BC = 180^\circ - D$, $AC = 180^\circ - E$, and $A = 180^\circ - EF$, $B = 180^\circ - FD$, $C = 180^\circ - DE$.



Affections of Right-angled Spherical Triangles.

35. The sides are of the same kind as their opposite angles; and conversely.

36. The hypotenuse is less or greater than 90° , according as a side and its adjacent angle, or the two sides, or the two angles, are like or unlike.

37. A side is less or greater than 90° , according as its adjacent angle and the hypotenuse, or the other side and the hypotenuse, are like or unlike.

38. An angle is acute or obtuse according as its adjacent side and the hypotenuse, or the other angle and the hypotenuse, are like or unlike.

Other Properties of Right-angled Spherical Triangles.

39. If the hypotenuse be 90° , one of the sides and its opposite angle will be 90° each; and the other side and angle will be of the same number of degrees.

40. If a side, or an angle, be 90° , the opposite angle, or side, and the hypotenuse, will be each 90° ; and the other side and angle will be of the same number of degrees.

41. If a side be less than the hypotenuse, their sum will be less than 180° ; and if it be greater than the hypotenuse, their sum will be greater than 180° .

42. If a side be less than its opposite angle, their sum will be less than 180° ; and if it be greater than its opposite angle, their sum will be greater than 180° .

43. The difference of the two oblique angles is less than 90° , and their sum is greater than 90° , and less than 270° .

44. Each of the three sides is either equal to or less than 90° ; or two of the sides are greater than 90° , and the third side is less.

45. A right-angled spherical triangle may have

1. One right angle, and two acute or two obtuse angles;
2. Or two right angles, and one acute or one obtuse angle;
3. Or all its angles right angles.

SPHERICAL TRIGONOMETRY.

46. **SPHERICAL** trigonometry is that science which treats of the analogies of the sides and angles of spherical triangles, and of the methods of computing the quantities of their sides and angles.

Let ABC (Plate, fig. 4) be a right-angled spherical triangle, having a right angle at B ; produce BC to a , and make $Ba = 90^\circ$; then a is the pole of AB (6).

From A as a pole describe the arc abD , meeting AB , AC (produced if necessary) in D and b . The triangle abC thus formed is called the *complemental* triangle to ABC . For Ca is the complement of CB , Cb is the complement of AC , the angle Cab is measured by BD , which is the complement of AB , and ab is the complement of bD , which measures the angle A . Also, the angle $aCb = ACB$ (22), and the angle abC is a right angle (21).

Because the sine and tangent of an arc or angle are the cosine and cotangent of its complement (23 Pl. Tr.), the sines and tangents of Ca , Cb , Cab , ba , are the cosines and cotangents of CB , AC , AB , and the angle A respectively; and the cosines and cotangents of the former are the sines and tangents of the latter.

47. Let ABC (plate, fig. 5) be a right-angled spherical triangle, having a right angle at B , and let O be the centre of the sphere. Draw OA , OB , OC . If we suppose the plane OAB to coincide with the plane of the paper, we must conceive the plane OBC to be perpendicular to OAB , because the angle at B is right.

Draw CF perpendicular to OB , and also Fm perpendicular to OB in the plane OAB ; then the angle CFm measures the inclination of the planes OBC , OBA (Def. 4. 2. Sup.). But these two planes are perpendicular to each other; therefore CFm is a right angle. Hence CF being perp. to the lines OFB and Fm , is perp. to the plane OBA (4. 2. Sup.).

Draw FE and BD perp. to OA , and join CE ; then CE will be perp. to AO . For CFE , CFO being right angles,

$CE^2 - EF^2 = CF^2 = CO^2 - OF^2$ (47. 1), therefore $CO^2 - CE^2 = OF^2 - EF^2 = OE^2$, therefore CEO is a right angle (48. 1). Now, since EC, EF are perp. to AO , the angle CEF is the inclination of the planes AOC, AOB (Def. 4. 2. Sup.), therefore the angle $CEF = CAB$ (23).

By construction CE is the sine, and OE is the cosine of the arc AC , CF is the sine, and OF is the cosine of CB , BD is the sine, and OD is the cosine of AB . Also, the sine, cosine, and tangent of the rectilinear angle CEF are the sine, cosine, and tangent of the spherical angle CAB .

PROP. I.

48. In a right angled spherical triangle ABC radius is to the sine of the hypotenuse AC , as the sine of either of the oblique angles A is to the sine of the opposite side BC . Plate, fig. 5.

In the rectilinear triangle CFE , having a right angle at F , $R : CE :: s. CEF : CF$ (49 Pl. Tr.). But the angle $CEF =$ spherical angle CAB (23.), and CE is the sine of the arc AC , and CF of BC . Hence $R : s. hyp. AC :: s. CAB : s. op. side BC$.

In the same manner it may be proved that $R : s. hyp. AC :: s. ACB : s. op. side AB$.

For if BC had been made the base, C would have been the angle at the base; consequently the angle ACB and the side AB would have entered the process, instead of the angle CAB and the side BC . Hence, in any proportion involving the angle A or C and the side BC or AB , the terms have similar relations, and may be substituted one for the other.

PROP. II.

49. In a right-angled spherical triangle ABC radius is to the sine of one side AB , as the tangent of the angle A adjacent to that side is to the tangent of the other side BC .

In the plane triangle CEF , having a right angle at F , $R : \tan. CEF \text{ or } CAB :: EF : FC$ (50 Pl. Tr.), and $\tan. BC : R :: FC : OF$ (41 Pl. Tr.);

therefore $\tan. BC : \tan. CAB :: EF : OF$ (22. 6).

But $EF : OF :: DB : OB$ (2. 6) $:: s. AB : R$.

Hence $s. AB : R :: \tan. BC : \tan. CAB$ (11. 5),

or $R : s. AB :: \tan. CAB : \tan. BC$ (A. 5).

In like manner it may be proved that

$$R : s. BC :: \tan. ACB : \tan. AB.$$

PROP. III.

50. In a right-angled spherical triangle ABC radius is to the tangent of either side AB , as the cotangent of the angle C opposite to that side is to the sine of the other side BC .

$$R : s. BC :: \tan. ACB : \tan. AB \text{ (49),}$$

and $\cot. ACB : R :: R : \tan. ACB$ (41 Pl. Tr.);

therefore $\cot. ACB : s. BC :: R : \tan. AB$ (22. 5),

$$\text{or } R : \tan. AB :: \cot. ACB : s. BC.$$

PROP. IV.

51. In a right-angled spherical triangle ABC radius is to the cosine of the hypotenuse AC , as the tangent of either of the oblique angles ACB is to the cotangent of the other angle BAC . Plate, fig. 4.

From the pole A describe the circle Db , meeting BC in a , and AC in b ; then is the triangle Cba the complement of the triangle CBA (46). Therefore in the triangle Cba , right-angled at b , Cb is the complement of AC , the hyp. of the triangle ABC ; ba is the comp. of the arc Db , the measure of the angle A ; aC , the hyp. of the triangle Cab , is the comp. of BC ; the arc BD , the measure of the angle a , is the comp. of AB .

Now in the triangle Cab , $R : s. Cb :: \tan. bCa : \tan. ba$ (49); that is, in the triangle ABC ,

$$R : \cos. AC :: \tan. ACB : \cot. BAC.$$

PROP. V.

52. In a right-angled spherical triangle ABC radius is to the cotangent of either of the oblique angles A , as the cotangent of the other angle ACB is to the cosine of the hypotenuse AC .

For $R : \cos. AC :: \tan. ACB : \cot. A$ (51),
 and $\cot. ACB : R :: R : \tan. ACB$ (41 Pl. Tr.);
 therefore $\cot. ACB : \cos. AC :: R : \cot. A$ (22. 5),
 or $R : \cot. A :: \cot. ACB : \cos. AC$.

PROP. VI.

53. In a right-angled spherical triangle ABC radius is to the tangent of the hypotenuse AC, as the cosine of either of the oblique angles A is to the tangent of the side AB adjacent to that angle.

The construction of prop. IV. remaining, in the complementary triangle Cba, $R : s. ba :: \tan. a : \tan. Cb$ (49). But $s. ba = \cot. A$, $\tan. a = \cot. AB$, $\tan. Cb = \cot. AC$. Hence $R : \cos. A :: \cot. AB : \cot. AC$.

But $\cot. AB : \cot. AC :: \tan. AC : \tan. AB$ (42 Pl. Tr.).

Hence $R : \cos. A :: \tan. AC : \tan. AB$ (11. 5),

or $R : \tan. AC :: \cos. A : \tan. AB$ (16. 5).

In the same manner it may be demonstrated that

$R : \tan. AC :: \cos. ACB : \tan. BC$. See remark on prop. I.

54. Cor. The tangents of any two arcs AB, AC, are reciprocally proportional to their cotangents.

For $\tan. AB : \tan. AC :: \cot. AC : \cot. AB$.

PROP. VII.

55. In a right-angled spherical triangle ABC radius is to the cotangent of the hypotenuse AC, as the tangent of either of the sides AB is to the cosine of the angle A adjacent to that side.

$R : \tan. AC :: \cos. A : \tan. AB$ (53),

and $R : \tan. AC :: \cot. AC : R$;

therefore $\cos. A : \tan. AB :: \cot. AC : R$ (11. 5),

or $R : \cot. AC :: \tan. AB : \cos. A$ (A. 5).

PROP. VIII.

56. In a right-angled spherical triangle ABC radius is to the cosine of either of the sides BC, as the cosine

of the other side AB is to the cosine of the hypotenuse AC .

In the comp. triangle abC , $R : s. Ca :: s. a : s. Cb$ (48), that is, $R : \cos. BC :: \cos. AB : \cos. AC$.

PROP. IX.

57. In a right-angled spherical triangle ABC radius is to the cosine of either of the sides BC , as the sine of the angle BCA adjacent to that side is to the cosine of the other angle A .

In the comp. triangle abC , $R : s. Ca :: s. aCb : s. ba$ (48), that is, $R : \cos. BC :: s. BCA : \cos. A$.

Note. By means of the preceding propositions all the cases of right-angled spherical triangles may be resolved.

PROP. X.

58. In all spherical triangles, whether right or oblique-angled, the sines of the sides are proportional to the sines of the angles opposite to them. Fig. 4 and 6.

Let ABC be either a right or an oblique-angled triangle; the sine of any side $BC : \text{sine of any other side } AC :: \text{sine of the angle } A \text{ opposite to } BC : \text{sine of the angle } B \text{ opposite to } AC$.

First, let ABC (fig. 4) be a right-angled triangle, having a right angle at B ; then

$$s. AC : R \text{ or } s. B :: s. BC : s. A \text{ (48),}$$

$$\text{and } s. AC : R :: s. AB : s. C;$$

$$\text{therefore } s. AB : s. C :: s. BC : s. A \text{ (11. 5).}$$

Secondly, let ABC (fig. 6) be an oblique-angled triangle. Through the angular point C draw an arc of a great circle CD perp. to the op. side AB .

In the right-angled triangle BCD , $s. BC : R :: s. CD : s. B$ (48), and in the right-angled triangle ADC , $s. AC : R :: s. CD : s. A$; therefore $s. BC : s. AC :: s. A : s. B$ (23. 5).

In the same manner it may be proved that $s. BC : s. AB :: s. A : s. C$.

If three of these quantities be given to find the fourth, the case will be doubtful, because the quantity is expressed by a sine, unless there be some other circumstances which may remove the ambiguity. (112 Pl. Tr.)

59. Cor. Hence $s. AB + s. BC : s. AB \sim s. BC ::$

$s. A + s. C : s. A \sim s. C$ (E. 5). Consequently

$\tan. \frac{1}{2} (AB + BC) : \tan. \frac{1}{2} (AB \sim BC) ::$

$\tan. \frac{1}{2} (A + C) : \tan. \frac{1}{2} (A \sim C)$ (125. Pl. Tr.)

60. *Remark 1.* In deducing formulæ for the solution of oblique-angled triangles, by drawing a perp. from the vertical angle upon the opposite side, if we suppose the perp. to fall within the triangle, and the sides and angles to be less than 90° , the formula so derived will answer in all other cases. But we must attend to the algebraic signs of the quantities (110, 111 Pl. Tr.); for if the signs of the terms be once true, and then be made to vary according to the variation of the signs of the factors which compose them, they will continue true. The reader must give attention to this circumstance in some of the following demonstrations.

61. *Remark 2.* If the sides of a spherical triangle be conceived to decrease indefinitely, the sines or the tangents of the sides will approach in length to the sides as their limit; and the spherical triangle will approach to a plane triangle as its limit. Therefore, in a rectilinear triangle, $AC : BC :: s. B : s. A$, as was proved in Plane Trig. In like manner, in all cases of spherical triangles, when the sines or the tangents of the sides enter into any analogy, the sides themselves may be substituted for their sines or tangents, and the conclusions will be true in all cases of rectilinear triangles.

PROP. XI.

62. In any spherical triangle ABC, if a perpendicular arc CD be drawn from any angle C upon the opposite side, or base, AB, the cosines of the segments of the base AD, DB will be proportional to the cosines of the adjacent sides AC, BC.

In the right-angled triangle ADC, $R : \cos. DC :: \cos. AD : \cos. AC$ (56), and in the right-angled triangle BDC, $R : \cos. DC :: \cos. BD : \cos. BC$; therefore $\cos. AD : \cos. AC :: \cos. BD : \cos. BC$, or $\cos. AD : \cos. BD :: \cos. AC : \cos. BC$.

PROP. XII.

63. The same construction remaining, the cosines of the angles A and B at the base will be proportional to the sines of the corresponding segments ACD, BCD of the vertical angle C.

In the triangle ACD, $R : \cos. CD :: s. ACD : \cos. A$ (57),
and in the triangle BCD, $R : \cos. CD :: s. BCD : \cos. B$;
therefore $s. ACD : \cos. A :: s. BCD : \cos. B$,
or $\cos. A : \cos. B :: s. ACD : s. BCD$.

PROP. XIII.

64. The same construction remaining, the sines of the segments of the base AD, BD will be reciprocally proportional to the tangents of the adjacent angles A and B at the base.

In the triangle ADC, $R : s. AD :: \tan. A : \tan. DC$ (49),
and in the triangle BDC, $R : s. BD :: \tan. B : \tan. DC$;
therefore $s. AD : s. BD :: \tan. B : \tan. A$.

PROP. XIV.

65. The same construction remaining, the cosines of the segments ACD, BCD of the vertical angle C will be reciprocally proportional to the tangents of their adjacent sides AC, BC.

In the triangle ADC, $R : \cos. ACD :: \tan. AC : \tan. CD$ (53),
and in the triangle BDC, $R : \cos. BCD :: \tan. BC : \tan. CD$;
therefore $\cos. ACD : \cos. BCD :: \tan. BC : \tan. AC$.

PROP. XV.

66. In any spherical triangle ABC, if a perpendicular arc CD be drawn from any angle C upon the opposite side, or base, AB, the sine of the sum of the sides containing the angle C is to the sine of their difference,

as the cotangent of half the sum of the segments of the vertical angle is to the tangent of half their difference.

Because $\tan. BC : \tan. AC :: \cos. ACD : \cos. BCD$ (65),
 $\tan. BC + \tan. AC : \tan. BC \smile \tan. AC :: \cos. BCD + \cos. ACD : \cos. BCD \smile \cos. ACD$ (E. 5). But $\tan. BC + \tan. AC : \tan. BC \smile \tan. AC :: s. (BC + AC) : s. (BC \smile AC)$ (126 Pl. Tr.), and $\cos. BCD + \cos. ACD : \cos. BCD \smile \cos. ACD :: \cot. \frac{1}{2} (BCD + ACD) : \tan. \frac{1}{2} (BCD \smile ACD)$ (125 Pl. Tr.). Hence $s. (BC + AC) : s. (BC \smile AC) :: \cot. \frac{1}{2} (BCD + ACD) \text{ or } \cot. \frac{1}{2} C : \tan. \frac{1}{2} (BCD \smile ACD)$ (11. 5).

67. Cor. Because $\frac{2 s. \frac{1}{2} (BC + AC) \times \cos. \frac{1}{2} (BC + AC)}{R} = s. (BC + AC)$ (115 Pl. Tr.), and $\frac{2 s. \frac{1}{2} (BC \smile AC) \times \cos. \frac{1}{2} (BC \smile AC)}{R} = s. (BC \smile AC)$,
 $s. \frac{1}{2} (BC + AC) \times \cos. \frac{1}{2} (BC + AC) : s. \frac{1}{2} (BC \smile AC) \times \cos. \frac{1}{2} (BC \smile AC) :: \cot. \frac{1}{2} C : \tan. \frac{1}{2} (BCD \smile ACD)$.

68. *Remark.* If two sides and the included angle of any spherical triangle be given, we can find, by this proposition, the term $BCD \smile ACD$, and consequently the angles at the vertex, ACD , BCD . Therefore when two sides and the included angle of a spherical triangle are given, the triangle may be resolved into two right-angled triangles, in each of which the hypotenuse and an angle are known; consequently the other angle and sides may be found by prop. IV and VI. Hence the side and angles of the proposed triangle will be determined.

PROP. XVI.

69. If a perpendicular arc CD be drawn from an angle C of a spherical triangle ABC , to the opposite side, or base, AB , the tangent of half the sum of the segments of the base will be to the tangent of half the sum of the other two sides, as the tangent of half their difference is to the tangent of half the difference of the segments of the base; that is, $\tan. \frac{1}{2} (BD + AD) : \tan. \frac{1}{2} (BC + AC) :: \tan. \frac{1}{2} (BC \smile AC) : \tan. \frac{1}{2} (BD \smile AD)$.

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Because $\cos. BC : \cos. AC :: \cos. BD : \cos. AD$ (62),
 $\cos. BC + \cos. AC : \cos. BC \sim \cos. AC :: \cos. BD + \cos. AD :$
 $\cos. BD \sim \cos. AD$ (E. 5). But $\cos. BC + \cos. AC : \cos. BC \sim$
 $\cos. AC :: \cot. \frac{1}{2} (BC + AC) : \tan. \frac{1}{2} (BC \sim AC)$ (125 Pl. Tr.);
 therefore $\cot. \frac{1}{2} (BC + AC) : \tan. \frac{1}{2} (BC \sim AC) :: \cot. \frac{1}{2} (BD +$
 $AD) : \tan. \frac{1}{2} (BD \sim AD)$ (11. 5), or $\cot. \frac{1}{2} (BC + AC) :$
 $\cot. \frac{1}{2} (BD + AD) :: \tan. \frac{1}{2} (BC \sim AC) : \tan. \frac{1}{2} (BD \sim AD)$.
 But $\cot. \frac{1}{2} (BC + AC) : \cot. \frac{1}{2} (BD + AD) :: \tan. \frac{1}{2} (BD + AD)$
 $\tan. \frac{1}{2} (BC + AC)$ (42 Pl. Tr.). Hence $\tan. \frac{1}{2} (BD + AD)$
 $: \tan. \frac{1}{2} (BC + AC) :: \tan. \frac{1}{2} (BC \sim AC) : \tan. \frac{1}{2} (BD \sim AD)$.

70. Cor. 1. When the perp. CD falls within the triangle,
 then $BD + AD = AB$ the base; and when it falls without,
 then $BD - AD = AB$ the base. Therefore, in the first case,
 the proportion becomes $\tan. \frac{1}{2} AB : \tan. \frac{1}{2} (BC + AC) ::$
 $\tan. \frac{1}{2} (BC - AC) : \tan. \frac{1}{2} (BD - AD)$; and, in the second
 case, it becomes, by inversion and alternation, $\tan. \frac{1}{2} AB :$
 $\tan. \frac{1}{2} (BC + AC) :: \tan. \frac{1}{2} (BC - AC) : \tan. \frac{1}{2} (BD + AD)$.

71. Cor. 2. From the demon. $\cot. \frac{1}{2} (BC + AC) : \tan.$
 $\frac{1}{2} (BC \sim AC) :: \cot. \frac{1}{2} (BD + AD) : \tan. \frac{1}{2} (BD \sim AD)$; that
 is, in any spherical triangle, the cot. of half the sum of two
 sides : tan. of half their difference :: cot. of half the sum of
 the segments of the base made by a perp. from the opposite
 angle : tan. of half their difference.

72. Remark. If the three sides of a spherical triangle be
 given, we can find, by the analogy in the proposition, or in
 cor. 2, the fourth term $AD \sim BD$ or $AD + BD$, and conse-
 quently the segments of the base AD, BD. Therefore when
 the three sides of a spherical triangle are given, the triangle
 may be resolved into two right-angled triangles, in each of
 which two sides are known; therefore the angles may be found
 by prop. VII, and consequently the angles of the proposed
 triangle will be determined.

These two theorems are ascribed to Napier, and are so well
 adapted to calculation by logarithms, that they are considered
 two of the most useful propositions in spherical trigonometry.

PROP. XVII.

73. In any spherical triangle, if a perpendicular arc
 be drawn from an angle to the opposite side, or base,

the cotangent of half the sum of the angles at the base will be to the tangent of half their difference, as the tangent of half the sum of the segments of the vertical angle is to the tangent of half their difference; that is, $\cot. \frac{1}{2} (A + B) : \tan. \frac{1}{2} (A \smile B) :: \tan. \frac{1}{2} (ACD + BCD) : \tan. \frac{1}{2} (ACD \smile BCD)$.

Because $\cos. A : \cos. B :: s. ACD : s. BCD$ (63),
 $\cos. A + \cos. B : \cos. A \smile \cos. B :: s. ACD + s. BCD :$
 $s. ACD \smile s. BCD$. But $\cos. A + \cos. B : \cos. A \smile \cos. B$
 $:: \cot. \frac{1}{2} (A + B) : \tan. \frac{1}{2} (A \smile B)$ (125 Pl. Tr.), and
 $s. ACD + s. BCD : s. ACD \smile s. BCD :: \tan. \frac{1}{2} (ACD +$
 $BCD) : \tan. \frac{1}{2} (ACD \smile BCD)$; therefore $\cot. \frac{1}{2} (A + B) :$
 $\tan. \frac{1}{2} (A \smile B) : \tan. \frac{1}{2} (ACD + BCD) : \tan. \frac{1}{2} (ACD \smile$
 $BCD)$ (11. 5) $:: \tan. \frac{1}{2} C : \tan. \frac{1}{2} (ACD \smile BCD)$ or ::
 $\tan. \frac{1}{2} (ACD + BCD) : \tan. \frac{1}{2} ACB$, according as the perp.
 CD falls within or without the triangle ABC.

74. Cor. Because $\cot. \frac{1}{2} (A + B) : \tan. \frac{1}{2} (ACD + BCD) ::$
 $\tan. \frac{1}{2} (A \smile B) : \tan. \frac{1}{2} (ACD \smile BCD)$, and $\cot. \frac{1}{2} (A + B) :$
 $\tan. \frac{1}{2} (ACD + BCD) :: \cot. \frac{1}{2} (ACD + BCD) : \tan. \frac{1}{2} (A + B)$
 (42 Pl. Tr.), $\cot. \frac{1}{2} (ACD + BCD) : \tan. \frac{1}{2} (A + B) :: \tan.$
 $\frac{1}{2} (A \smile B) : \tan. \frac{1}{2} (ACD \smile BCD)$ (11. 5); that is, if a perpen-
 dicular arc be drawn from an angle of a spherical triangle to
 the opposite side, the cot. of half the sum of the segments of
 the vertical angle : tan. of half the sum of the angles at the
 base :: tan. of half their difference : tan. of half the difference
 of the segments of the vertical angle.

75. Remark. If the three angles of a spherical triangle be
 given, we can find, by the analogy in the prop. or cor. the
 term $ACD \smile BCD$ or $ACD + BCD$, and consequently the
 angles at the vertex, ACD , BCD . Therefore when the three
 angles of a spherical triangle are given, the triangle may be
 resolved into two right-angled triangles, in each of which the
 angles are known; therefore the sides may be found by prop.
 IV, and consequently the sides of the proposed triangle will
 be determined.

These two theorems are very commodious in practice.

PROP. XVIII.

76. In any spherical triangle the sine of half the sum of any two sides is to the sine of half their difference, as the cotangent of half the included angle is to the tangent of half the difference of the other two angles.

Let the two sides AC, BC, and the included angle C be proposed; and draw CD perp. to AB. Then (59) $\tan. \frac{1}{2} (AC + BC) : \tan. \frac{1}{2} (AC \smile BC) :: \tan. \frac{1}{2} (A + B) : \tan. \frac{1}{2} (A \smile B)$. For $\tan. \frac{1}{2} (A + B)$ substitute its value

$$\frac{\cot. \frac{1}{2} C \times \tan. \frac{1}{2} (ACD \smile BCD)}{\tan. \frac{1}{2} (A \smile B)} \quad (74), \text{ then } \tan. \frac{1}{2} (AC + BC) :$$

$\tan. \frac{1}{2} (AC \smile BC) :: \cot. \frac{1}{2} C \times \tan. \frac{1}{2} (ACD \smile BCD) : \tan.^2 \frac{1}{2} (A \smile B)$. By art. 67 s. $\frac{1}{2} (AC + BC) \times \cos. \frac{1}{2} (AC + BC) : s. \frac{1}{2} (AC \smile BC) \times \cos. \frac{1}{2} (AC \smile BC) :: \cot. \frac{1}{2} C : \tan. \frac{1}{2} (ACD \smile BCD)$. Multiply together the corresponding terms of these two proportions, and for $\cos. \times \tan.$ substitute its equal, $\sin. \times R$; then $\sin.^2 \frac{1}{2} (AC + BC) : \sin.^2 \frac{1}{2} (AC \smile BC) :: \cot.^2 \frac{1}{2} C : \tan.^2 \frac{1}{2} (A \smile B)$, therefore $s. \frac{1}{2} (AC + BC) : s. \frac{1}{2} (AC \smile BC) :: \cot. \frac{1}{2} C : \tan. \frac{1}{2} (A \smile B)$.

77. Cor. From the first analogy in the demonstr. we obtain

$$\tan. \frac{1}{2} (A \smile B) = \tan. \frac{1}{2} (A + B) \times \frac{\tan. \frac{1}{2} (AC \smile BC)}{\tan. \frac{1}{2} (AC + BC)}, \text{ and from}$$

$$\text{the last, } \tan. \frac{1}{2} (A \smile B) = \cot. \frac{1}{2} C \times \frac{s. \frac{1}{2} (AC \smile BC)}{s. \frac{1}{2} (AC + BC)};$$

$$\text{therefore } \tan. \frac{1}{2} (A + B) \times \frac{\tan. \frac{1}{2} (AC \smile BC)}{\tan. \frac{1}{2} (AC + BC)} =$$

$$\cot. \frac{1}{2} C \times \frac{s. \frac{1}{2} (AC \smile BC)}{s. \frac{1}{2} (AC + BC)}, \text{ therefore}$$

$$\tan. \frac{1}{2} (A + B) = \cot. \frac{1}{2} C \times \frac{\cos. \frac{1}{2} (AC \smile BC)}{\cos. \frac{1}{2} (AC + BC)}, \text{ therefore}$$

$\cos. \frac{1}{2} (AC + BC) : \cos. \frac{1}{2} (AC \smile BC) :: \cot. \frac{1}{2} C : \tan. \frac{1}{2} (A + B)$, that is, in any spherical triangle the cosine of half the sum of any two sides is to the cosine of half their difference, as the cotangent of half the included angle is to the tangent of half the sum of the other two angles.

78. Remark. From this proposition and cor. we obtain half the sum S, and half the difference D, of the two angles A and B; therefore $S + D$ and $S - D$ are the two angles.

These two elegant theorems were invented by Lord Napier, and are better adapted to the logarithmic solution of a spherical triangle than any other, when two sides and the included angle are given. They likewise possess the advantage of being easily remembered.

PROP. XIX.

79. In any spherical triangle the sine of half the sum of any two angles is to the sine of half their difference, as the tangent of half the included side is to the tangent of half the difference of the other two sides: also, the cosine of half the sum of any two angles is to the cosine of half their difference, as the tangent of half the included side is to the tangent of half the sum of the other two sides. Fig. page 80.

Let the angles C; CAB, and the included side AC, be proposed; then in the supplemental triangle FDE we have the sides DE, EF, and the included angle E (34). Hence $s. \frac{1}{2} (DE + EF) : s. \frac{1}{2} (DE - EF) :: \cot. \frac{1}{2} E : \tan. \frac{1}{2} (D - F)$ (76), and $\cos. \frac{1}{2} (DE + EF) : \cos. \frac{1}{2} (DE - EF) :: \cot. \frac{1}{2} E : \tan. \frac{1}{2} (D + F)$ (77).

Now $C = 180^\circ - DE$, $A = 180^\circ - EF$, $AC = 180^\circ - E$, $CB = 180^\circ - D$, $AB = 180^\circ - F$. Hence sine or cos. $\frac{1}{2} (A + C) = \text{sine or cos. } \frac{1}{2} (360^\circ - DE - EF) = \text{sine or cos. } \frac{1}{2} (DE + EF)$ (34 and 35 Pl. Tr.); sine or cos. $\frac{1}{2} (A - C) = \text{sine or cos. } \frac{1}{2} (DE - EF)$; $\tan. \frac{1}{2} AC = \tan. \frac{1}{2} (180^\circ - E) = \tan. (90^\circ - \frac{1}{2} E) = \cot. \frac{1}{2} E$ (23 Pl. Tr.); $\tan. \frac{1}{2} (AB - BC) = \tan. \frac{1}{2} (D - F)$; $\tan. \frac{1}{2} (AB + BC) = \tan. \frac{1}{2} (360^\circ - D - F) = \tan. \frac{1}{2} (D + F)$ (36 Pl. Tr.).

Hence we obtain, by substitution,
 $s. \frac{1}{2} (A + C) : s. \frac{1}{2} (A - C) :: \tan. \frac{1}{2} AC : \tan. \frac{1}{2} (AB - BC)$,
 and $\cos. \frac{1}{2} (A + C) : \cos. \frac{1}{2} (A - C) :: \tan. \frac{1}{2} AC : \tan. \frac{1}{2} (AB + BC)$.

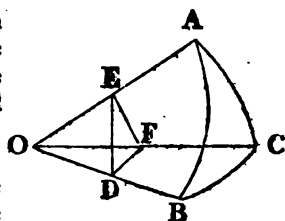
80. *Remark.* By this proposition we can find half the sum S, and half the difference D, of the two sides AB, BC of a spherical triangle; therefore $S + D = \text{greater side}$, and $S - D = \text{less side}$.

These two elegant theorems were invented by Napier, and are more convenient in practice than any other, when two angles and the included side of a spherical triangle are given.

PROP. XX.

81. The rectangle of the sines of any two sides of a spherical triangle is to the radius, as the rectangle of the radius and the cosine of the other side — the rectangle of the cosines of the same two sides, is to the cosine of the angle included by those sides.

Join OA, OB, OC; draw FD in the plane OBC, and DE in the plane OAB, each perpendicular to OB the common section of the planes; and join E, F.



Because the angle EDF is the measure of the inclination of the planes OBC, OAB, it is also the measure of the spherical angle ABC (23). Now

$$\cos. EDF = \frac{R(DE^2 + DF^2 - EF^2)}{2 DE \cdot DF} \quad (63 \text{ Pl. Tr.}), \text{ and}$$

$$\cos. EOF = \frac{R(OE^2 + OF^2 - EF^2)}{2 OE \cdot OF} \quad \text{From the latter equation}$$

$$\text{we deduce } EF^2 = OE^2 + OF^2 - \frac{2 OE \cdot OF \cos. EOF}{R},$$

$$\text{which substituted in the former equation produces } \cos. EDF = \frac{R(DE^2 + DF^2 - OE^2 - OF^2) + 2 OE \cdot OF \cos. EOF}{2 DE \cdot DF}.$$

$$\text{Now } OE^2 - ED^2 = OD^2, \text{ and } OF^2 - DF^2 = OD^2 \quad (47. 1);$$

$$\text{therefore } \cos. EDF = \cos. ABC =$$

$$\frac{OE \cdot OF \cos. EOF - R \times OD^2}{DE \cdot DF} =$$

$$\frac{OE \cdot OF \cos. AC - R \times OD^2}{DE \cdot DF}.$$

$$\text{Because } \frac{OE}{DE} = \frac{R}{s. DOE} \quad (49 \text{ Pl. Tr.}) = \frac{R}{s. AB}, \text{ and,}$$

$$\frac{OF}{DF} = \frac{R}{s. DOF} = \frac{R}{s. BC}, \text{ and } \frac{OD}{DE} = \frac{\cos. DOE}{s. DOE} = \frac{\cos. AB}{s. AB},$$

and $\frac{OD}{DF} = \frac{\cos. DOF}{s. DOF} = \frac{\cos. BC}{s. BC}$; if these values be substituted

in the last value of $\cos. ABC$, we shall obtain

$$\cos. B = \frac{R^2 \cos. AC - R \cos. AB \cos. BC}{s. AB s. BC}, \text{ consequently}$$

$$s. AB \times s. BC : R :: R \cos. AC - \cos. AB \times \cos. BC : \cos. B.$$

82. *Remark 1.* This theorem resolves immediately the case in which it is required to find an angle by means of the three sides of a triangle, but is not adapted to calculation by logarithms. Therefore the natural sines of the given sides must be employed in the calculation, which may often require a tedious multiplication and division.

This proposition being applied successively to all the angles of a spherical triangle, furnishes three equations which are sufficient for the resolution of all the cases of spherical trigonometry. It has the same generality with respect to spherical triangles as the theorem in article 63 Pl. Tr. has with respect to rectilinear triangles.

83. *Remark 2.* If the three angles of the triangle ABC be denoted by A, B, C , and their corresponding opposite sides by a, b, c ; then the preceding theorem gives

$$\cos. B = \frac{R^2 \cos. b - R \cos. a \cos. c}{\sin a \sin c}.$$

A similar formula may be readily obtained for the cosines of any of the sides of a spherical triangle in terms of the sines and cosines of the three angles. For since any spherical triangle, whose three angles are A, B, C , and three opposite sides a, b, c , answers to the polar triangle, whose three sides are $180^\circ - A, 180^\circ - B, 180^\circ - C$, and opposite angles $180^\circ - a, 180^\circ - b, 180^\circ - c$ (34), we deduce from the last theorem, $\cos. (180^\circ - a) =$

$$\frac{R^2 \cos. (180^\circ - A) - R \cos. (180^\circ - B) \cos. (180^\circ - C)}{\sin (180^\circ - B) \times \sin (180^\circ - C)}.$$

Because $\cos. (180^\circ - a) = -\cos. a$ (35 Pl. Tr.), and $\cos. (180^\circ - A) = -\cos. A$, &c. this equation, when reduced,

$$\text{becomes } \cos. a = \frac{R^2 \cos. A + R \cos. B \cos. C}{\sin B \sin C}.$$

This formula resolves immediately the case in which it is required to find a side of a triangle by means of the three angles, but is not adapted to calculation by logarithms.

PROP. XXI.

84. In any spherical triangle, the rectangle under the sines of any two sides is to the sine of $\frac{1}{2}$ (the third side + the difference between the other two sides) $\times$ the sine of $\frac{1}{2}$ (the third side — the difference between the other two sides), as the square of radius is to the square of the sine of half the angle included by the first two sides.

If the value of the cosine of C (82) be substituted in the formula $2 \operatorname{sine}^2 \frac{1}{2} C = r^2 - r \cos. C$ (46, no. 3, Pl. Tr.),

$$\text{then } \frac{2 \operatorname{sine}^2 \frac{1}{2} C}{r^2} = 1 - \frac{\cos. C}{r} =$$

$$1 - \frac{r \cos. c - \cos. a \cos. b}{\operatorname{sine} a \operatorname{sine} b} =$$

$$\frac{\operatorname{sine} a \operatorname{sine} b - r \cos. c + \cos. a \cos. b}{\operatorname{sine} a \operatorname{sine} b} =$$

$$\frac{\cos. a \cos. b + \operatorname{sine} a \operatorname{sine} b - r \cos. c}{\operatorname{sine} a \operatorname{sine} b}.$$

But $\cos. a \cos. b + \operatorname{sine} a \operatorname{sine} b = r \cos. (a - b)$ (122 Pl. Tr.).

Consequently $\frac{2 \operatorname{sine}^2 \frac{1}{2} C}{r^2} = \frac{r \cos. (a - b) - r \cos. c}{\operatorname{sine} a \operatorname{sine} b}$. But

$r \cos. (a - b) - r \cos. c = 2 s. \frac{1}{2} (c + b - a) s. \frac{1}{2} (c + a - b)$ (123 Pl. Tr.). Hence

$$\frac{\operatorname{sine}^2 \frac{1}{2} C}{r^2} = \frac{s. \frac{1}{2} (c + b - a) s. \frac{1}{2} (c + a - b)}{\operatorname{sine} a \operatorname{sine} b}, \text{ therefore}$$

$$\operatorname{sine} a \operatorname{sine} b : \operatorname{sine} \frac{1}{2} (c + b - a) \operatorname{sine} \frac{1}{2} (c + a - b) :: r^2 : \operatorname{sine}^2 \frac{1}{2} C.$$

$$85. \text{ Cor. Hence } \operatorname{sine} \frac{1}{2} C = r \sqrt{\frac{s. \frac{1}{2} (c + b - a) s. \frac{1}{2} (c + a - b)}{\operatorname{sine} a \times \operatorname{sine} b}},$$

which is a convenient logarithmic formula for the value of the sine of half an angle, when the three sides of a spherical triangle are known.

PROP. XXII.

85. In any spherical triangle, the rectangle under the sines of any two sides is to the sine of $\frac{1}{2}$ (the sum of the same two sides + the third side) $\times$ the sine of $\frac{1}{2}$ (the sum of the same two sides — the third side), as the square of radius is to the square of the cosine of half the angle included by the first two sides.

If the value of the cosine of C (83) be substituted in the formula

$$2 \cos.^2 \frac{1}{2} C = r^2 + r \cos. C, * \text{ then } \frac{2 \cos.^2 \frac{1}{2} C}{r^2} = 1 + \frac{\cos. C}{r} =$$

$$\frac{r \cos. c - \cos. a \cos. b + \text{sine } a \text{ sine } b}{\text{sine } a \text{ sine } b}.$$

Because $r \cos. (a+b) = \cos. a \cos. b - \text{sine } a \text{ sine } b$ (122 Pl. Tr.), $-r \cos. (a+b) = -\cos. a \cos. b + \text{sine } a \text{ sine } b$.

$$\text{Hence } \frac{2 \cos.^2 \frac{1}{2} C}{r^2} = \frac{r \cos. c - r \cos. (a+b)}{\text{sine } a \text{ sine } b}.$$

But $r \cos. c - r \cos. (a+b) = 2 s. \frac{1}{2} (a+b-c) s. \frac{1}{2} (a+b+c)$ (123 Pl. Tr.). Consequently

$$\frac{\cos.^2 \frac{1}{2} C}{r^2} = \frac{s. \frac{1}{2} (a+b+c) s. \frac{1}{2} (a+b-c)}{\text{sine } a \text{ sine } b},$$

therefore $\text{sine } a \times \text{sine } b : \text{sine } \frac{1}{2} (a+b+c) \text{ sine } \frac{1}{2} (a+b-c) :: r^2 : \cos.^2 \frac{1}{2} C$.

$$86. \text{ Cor. 1. } \cos. \frac{1}{2} C = r \sqrt{\frac{s. \frac{1}{2} (a+b+c) s. \frac{1}{2} (a+b-c)}{\text{sine } a \text{ sine } b}},$$

which is a convenient log. formula for the value of the cosine of half an angle, when the three sides of a spherical triangle are known.

$$87. \text{ Cor. 2. Because } \frac{r \text{ sine } \frac{1}{2} C}{\cos. \frac{1}{2} C} = \tan. \frac{1}{2} C \text{ (41 Pl. Tr.),}$$

it follows from art. 84 and 86 that

$$\tan. \frac{1}{2} C = r \sqrt{\frac{s. \frac{1}{2} (c+b-a) s. \frac{1}{2} (c+a-b)}{s. \frac{1}{2} (a+b+c) s. \frac{1}{2} (a+b-c)}}.$$

* By similar triangles ACL, BDF (fig. p. 10), $CA : CL :: BD$ or $2 CL : DF$, that is, radius : cos. any arc $\frac{1}{2} C :: 2 \cos. \frac{1}{2} C : \text{versed sine } C$, therefore $2 \cos.^2 \frac{1}{2} C = r^2 + r \cos. C$.

88. Now each of these three formulæ, or values of the sine, cosine, and tan. of $\frac{1}{2} C$, will determine an angle of any spherical triangle, when the three sides are given; and is well adapted to logarithmic calculation.

By the same method of investigation logarithmic formulæ for determining each of the sides of a spherical triangle, in terms of the three angles, may be obtained.

PROP. XXIII.

89. In any spherical triangle, the rectangle under the sines of any two angles is to the cosine of $\frac{1}{2}$ (the sum of the same two angles + the third angle) $\times$ cosine of $\frac{1}{2}$ (the sum of the same two angles — the third angle), as the square of radius is to the square of half the side contained between the first two angles.

If the value of the cosine of a (83) be substituted in the formula $\frac{2 \sin^2 \frac{1}{2} a}{r^2} = 1 - \frac{\cos. a}{r}$ (46, no. 3 Pl. Tr.),

$$\text{then } \frac{2 \sin^2 \frac{1}{2} a}{r^2} = \frac{\sin B \sin C - \cos. B \cos. C - r \cos. A}{\sin B \sin C}.$$

But $\sin B \sin C - \cos. B \cos. C = -r \cos. (B + C)$ (122 Pl. Tr.). Consequently

$$\frac{2 \sin^2 \frac{1}{2} a}{r^2} = \frac{-r \cos. (B + C) - r \cos. A}{\sin B \sin C}.$$

But $r \cos. (B + C) + r \cos. A = 2 \cos. \frac{1}{2} (A + B + C) \times \cos. \frac{1}{2} (B + C - A)$ (123 Pl. Tr.). Hence $\frac{\sin^2 \frac{1}{2} a}{r^2} =$

$$-\frac{\cos. \frac{1}{2} (A + B + C) \cos. \frac{1}{2} (B + C - A)}{\sin B \sin C}, \text{ therefore}$$

$$\sin B \sin C : -\cos. \frac{1}{2} (A + B + C) \cos. \frac{1}{2} (B + C - A)$$

$$:: r^2 : \sin^2 \frac{1}{2} a.$$

$$90. \text{ Cor. } \sin \frac{1}{2} a = r \sqrt{\frac{-\cos. \frac{1}{2} (A + B + C) \cos. \frac{1}{2} (B + C - A)}{\sin B \sin C}}.$$

Remark. The second member of this equation is always positive, though under a negative form; for $\frac{1}{2} (A + B + C)$ is greater than 90° (29), therefore its cosine will be negative (35 Pl. Tr.); consequently the expression $-\cos. \frac{1}{2} (A + B + C)$

the sine of

will become positive. Again, $B + C - A$ cannot exceed 180° (25); or $\frac{1}{2}(B + C - A)$ cannot exceed 90° ; therefore $\cos. \frac{1}{2}(B + C - A)$ must also be positive.

PROP. XXIV.

91. In any spherical triangle, the rectangle under the sines of any two angles is to the cosine of $\frac{1}{2}$ (the third angle + the difference between the other two angles) $\times$ the cosine of $\frac{1}{2}$ (the third angle — the difference between the other two angles), as the square of radius is to the square of the cosine of half the side contained between the first two angles.

If the value of the cosine of a (83) be substituted in the formula

$$\frac{2 \cos.^2 \frac{1}{2} a}{r^2} = 1 + \frac{\cos. a}{r}, \text{ then } \frac{2 \cos.^2 \frac{1}{2} a}{r^2} =$$

$$\frac{\text{sine } B \text{ sine } C + \cos. B \cos. C + r \cos. A}{\text{sine } B \text{ sine } C}. \text{ But sine } B \text{ sine } C +$$

$\cos. B \cos. C = r \cos. (B - C)$ (122 Pl. Tr.). Consequently

$$\frac{2 \cos.^2 \frac{1}{2} a}{r^2} = \frac{r \cos. (B - C) + r \cos. A}{\text{sine } B \text{ sine } C}. \text{ But } r \cos. (B - C) +$$

$$r \cos. A = 2 \cos. \frac{1}{2} (A + B - C) \cos. \frac{1}{2} (A + C - B)$$

$$(123 \text{ Pl. Tr.}). \text{ Hence } \frac{\cos.^2 \frac{1}{2} a}{r^2} =$$

$$\frac{\cos. \frac{1}{2} (A + B - C) \cos. \frac{1}{2} (A + C - B)}{\text{sine } B \text{ sine } C}, \text{ therefore}$$

$$\text{sine } B \text{ sine } C : \cos. \frac{1}{2} (A + B - C) \cos. \frac{1}{2} (A + C - B) :: r^2 : \cos.^2 \frac{1}{2} a.$$

$$92. \text{ Cor. 1. } \cos. \frac{1}{2} a = r \sqrt{\frac{\cos. \frac{1}{2} (A + B - C) \cos. \frac{1}{2} (A + C - B)}{\text{sine } B \text{ sine } C}}.$$

$$93. \text{ Cor. 2. Because } \frac{r \text{ sine } \frac{1}{2} a}{\cos. \frac{1}{2} a} = \tan. \frac{1}{2} a, \text{ it follows from art. 90}$$

$$\text{and 92, that } \tan. \frac{1}{2} a = r \sqrt{\frac{-\cos. \frac{1}{2} (B + C + A) \cos. \frac{1}{2} (B + C - A)}{\cos. \frac{1}{2} (A + B - C) \cos. \frac{1}{2} (A + C - B)}}.$$

Each of these three formulæ will determine a side of a spherical triangle, when the three angles are given, and is well adapted to logarithmic calculation.

94. *Remark.* Prop. 21 is not proper to be used when the angle sought is near 180° , because the difference of the logarithmic sines of $\frac{1}{2} C$ for $1''$ is then very small; and *very* near 180° the log. sines of $\frac{1}{2} C$ (in the tables) are the same for arcs which differ by many seconds. In this case therefore it will be proper to use Prop. 22. Prop. 21 is most proper to be used when C is less than 90° , for then there is a considerable difference between the log. sines of $\frac{1}{2} C$ for $1''$. The reason of the very small variation of the sines of arcs near 90° is, that the logarithms are not continued to more than seven decimals.

Prop. 22 is not proper to be used when the angle sought is very small, because the difference between the log. cosines of $\frac{1}{2} C$ for $1''$ is then very small; and *very* near 0° the log. cosines of $\frac{1}{2} C$ are the same for arcs which differ by many seconds. Therefore when an arc is extremely small, Prop. 21 must be used. Prop. 22 is most proper to be used when C is between 90° and 180° , for then there is a considerable difference between the log. cosines of $\frac{1}{2} C$ for $1''$ (86 Pl. Tr.).

Prop. 23 is not proper to be used, when the arc a is near 180° , but is most proper to be used when a is less than 90° . See observation on Prop. 21.

Prop. 24 is not proper to be used when the arc a is near 0° , but is most proper to be used when a is between 90° and 180° . See observation on Prop. 22.

Formulæ for the Solution of Spherical Triangles.

95. The following formulæ are sufficient for the logarithmic solution of all the cases of a right-angled spherical triangle.

$$r \sin a = \sin c \sin A = \tan. b \cot. B$$

$$r \cos. B = \cos. b \sin A = \tan. a \cot. c$$

$$r \cos. c = \cos. a \cos. b = \cot. A \cot. B$$

These forms will also hold in any case, by taking such other sides and angles as are similarly situated with respect to one another.

96. The following formulæ are sufficient for the logarithmic solution of all the cases of any spherical triangle. But there are other simple formulæ which are better adapted to the solution of right-angled triangles. See articles 95 and 101.

1. Given two sides a, c , and an angle C opposite to one of them, to find the other opposite angle A .

$$\text{Sine } A = \frac{\text{sine } a \text{ sine } C}{\text{sine } c}.$$

2. Given two angles A, C , and a side c opposite to one of them, to find the other opposite side a .

$$\text{Sine } a = \frac{\text{sine } c \text{ sine } A}{\text{sine } C}.$$

3. Given two sides b, c , and the included angle A , to find the other angles B, C .

$$\text{Tan. } \frac{1}{2} (B + C) = \frac{\cos. \frac{1}{2} (b - c)}{\cos. \frac{1}{2} (b + c)} \cot. \frac{1}{2} A.$$

$$\text{Tan. } \frac{1}{2} (B - C) = \frac{\text{sine } \frac{1}{2} (b - c)}{\text{sine } \frac{1}{2} (b + c)} \cot. \frac{1}{2} A.$$

4. Given two angles B, C , and the side between them a , to find the other sides b, c .

$$\text{Tan. } \frac{1}{2} (b + c) = \frac{\cos. \frac{1}{2} (B - C)}{\cos. \frac{1}{2} (B + C)} \tan. \frac{1}{2} a.$$

$$\text{Tan. } \frac{1}{2} (b - c) = \frac{\text{sine } \frac{1}{2} (B - C)}{\text{sine } \frac{1}{2} (B + C)} \tan. \frac{1}{2} a.$$

5. Given the three sides a, b, c , to find one of the angles A .

$$\text{Tan. } \frac{1}{2} A = \sqrt{\frac{\text{sine } \frac{1}{2} (a + b - c) \text{ sine } \frac{1}{2} (a + c - b)}{\text{sine } \frac{1}{2} (b + c - a) \text{ sine } \frac{1}{2} (a + b + c)}}.$$

6. Given the three angles A, B, C , to find one of the sides a .

$$\text{Tan. } \frac{1}{2} a = \sqrt{\frac{-\cos. \frac{1}{2} (B + C - A) \cos. \frac{1}{2} (A + B + C)}{\cos. \frac{1}{2} (A + B - C) \cos. \frac{1}{2} (A + C - B)}}.$$

Instead of the last two formulæ the four following are often used.

$$\text{Sine } \frac{1}{2} A = \sqrt{\frac{\text{sine } \frac{1}{2} (a + b - c) \text{sine } \frac{1}{2} (a + c - b)}{\text{sine } b \text{sine } c}}.$$

$$\text{Sine } \frac{1}{2} a = \sqrt{\frac{-\cos. \frac{1}{2} (A + B + C) \cos. \frac{1}{2} (B + C - A)}{\text{sine } B \text{sine } C}}.$$

$$\text{Cos. } \frac{1}{2} A = \sqrt{\frac{\text{sine } \frac{1}{2} (a + b + c) \text{sine } \frac{1}{2} (b + c - a)}{\text{sine } b \text{sine } c}}.$$

$$\text{Cos. } \frac{1}{2} a = \sqrt{\frac{\cos. \frac{1}{2} (A + B - C) \cos. \frac{1}{2} (A + C - B)}{\text{sine } B \text{sine } C}}.$$

It is manifest that the preceding equations are susceptible of many changes, by which they become applicable to the solution of cases when different parts are given. Thus, case 3 becomes, by transposition,

$$\text{Cot. } \frac{1}{2} A = \frac{\text{sine } \frac{1}{2} (b + c)}{\text{sine } \frac{1}{2} (b - c)} \tan. \frac{1}{2} (B - C) = \frac{\cos. \frac{1}{2} (b + c)}{\cos. \frac{1}{2} (b - c)}$$

$$\tan. \frac{1}{2} (B + C);$$

by which equation we can find the angle, when the two sides which include it and the other two angles are given.

97. The formulæ in the last four cases deserve particular notice both for their elegance, and for their property of showing whether the arc or angle which they express is less or greater than a quadrant or 90° . For since the sine of an arc is the same as the sine of its supplement, both in value (34 Pl. Tr.) and in sign (97 Pl. Tr.), it is impossible to distinguish whether that arc ought to be less or greater than a quadrant. But when we know the cosine, the tangent, or the cotangent of an arc, and also know that the arc cannot be equal to a semi-circle (25), which is the limit of the sides of spherical triangles, and of the arcs which measure their angles, we perceive from the sign of the result whether the arc sought is less or greater than a quadrant. In the first case the cosine, the tangent, and the cotangent have the sign +, in the second the sign — (110 Pl. Tr.). If therefore we be careful to give to the known quantities which enter into the preceding formulæ, the signs which ought to affect them according to the values of the arcs to which those signs belong, the sign of the result will always determine the *species* of the arc or angle required; that is, whether the arc is less or greater than a quadrant, and whether the angle is acute or obtuse.

Napier's Rules of the Circular Parts.

The two rules invented by Lord Napier are very useful in spherical trigonometry, because they reduce all the theorems employed in the solution of right-angled triangles to two, which are simple and easy to remember, and very convenient in practice.

98. *Def. 1.* In a right-angled spherical triangle, if we omit the right angle, and consider only the five remaining parts, or the three sides and the two oblique angles, then the two sides containing the right angle, and the complements of the two oblique angles and of the hypotenuse, are called the *circular parts*. See plate, fig. 4.

Thus, in the triangle ABC , having a right angle at B , the circular parts are AB , BC , with the complements of C , AC , and A . These parts are called circular, because, when they are named in the natural order of their succession, they go round the circle.

99. *Def. 2.* Of the five circular parts if any one be taken for the middle part, the two which are contiguous to it, on the right and left, are called the *adjacent parts*; and the other two, each of which is separated from the middle part by an adjacent part, are called *opposite parts*. By some authors they are called *adjacent extremes*, and *opposite extremes*.

Thus, in the right-angled triangle ABC , AB , BC , $90^\circ - C$, $90^\circ - AC$, $90^\circ - A$, are the circular parts (*Def. 1*); and if any one of them, as AB , be reckoned the middle part, then BC and $90^\circ - A$, which are contiguous to AB , on different sides, are called adjacent parts; and $90^\circ - C$ and $90^\circ - AC$, which are separated from it, are called opposite parts. If BC be taken for the middle part, then AB and $90^\circ - C$ are the adjacent parts; and $90^\circ - AC$ and $90^\circ - A$ are the opposite parts. If $90^\circ - AC$ be the middle part, then $90^\circ - C$ and $90^\circ - A$ are adjacent parts, and AB and BC are opposite parts.

This arrangement being made, the two rules of the circular parts are contained in the following proposition.

100. In a right-angled spherical triangle, the rectangle under the radius and the sine of the middle part is equal to the rectangle under the tangents of the adja-

cent parts, or to the rectangle under the cosines of the opposite parts.

Note. The reader must remember, that the cosine of the complement of an arc or an angle is the sine of the arc or angle; and that the tangent of the comp. is the cotangent; and conversely. The demonstration is as follows.

Cases 1, 2. Make AB the middle part.

Sine AB : tan. BC :: R : tan. A (49) :: cot. A : R,
therefore $R \times \text{sine AB} = \text{cot. A} \times \text{tan. BC}$.

Again, R : s. C :: s. AC : s. AB (48),
therefore $R \times \text{s. AB} = \text{s. C} \times \text{s. AC}$.

If BC be made the middle part, the prop. may be proved in the same manner.

Cases 3, 4. Make the compl. of the angle A the middle part.

In the triangle abC , tan. bC : s. ab :: tan. a : R (49), or
cot. AC : cos. A :: cot. AB : R (46) :: R : tan. AB,
therefore $R \times \text{cos. A} = \text{cot. AC} \times \text{tan. AB}$.

Again, R : s. aC :: s. C : s. ab (48),
or R : cos. BC :: s. C : cos. A (46),
therefore $R \times \text{cos. A} = \text{s. C} \times \text{cos. BC}$.

If the compl. of C be made the middle part, the prop. may be proved in the same manner.

Case 5. Make the compl. of AC the middle part.

In the triangle abC , s. bC : tan. ab :: R : tan. C (49) ::
cot. C : R, or cos. AC : cot. A :: cot. C : R (46), therefore
 $R \times \text{cos. AC} = \text{cot. A} \times \text{cot. C}$.

Again, R : s. aC :: s. a : s. bC , (48) or R : cos. BC :: cos. AB :
cos. AC, therefore $R \times \text{cos. AC} = \text{cos. AB} \times \text{cos. BC}$.

Hence the proposition is manifest.

101. The following table contains all the equations arising from these theorems, by means of which all the cases of right-angled spherical triangles may be resolved with facility.

Mid. Part	Equation
1. AB	$R \times s. AB = \begin{cases} \cot. A \times \tan. BC \\ s. C \times s. AC \end{cases}$
2. BC	$R \times s. BC = \begin{cases} \cot. C \times \tan. AB \\ s. A \times s. AC \end{cases}$
3. comp. A	$R \times \cos. A = \begin{cases} \cot. AC \times \tan. AB \\ s. C \times \cos. BC \end{cases}$
4. comp. C	$R \times \cos. C = \begin{cases} \cot. AC \times \tan. BC \\ s. A \times \cos. AB \end{cases}$
5. comp. AC	$R \times \cos. AC = \begin{cases} \cot. C \times \cot. A \\ \cos. AB \times \cos. BC \end{cases}$

102. To apply the general prop. to the resolution of any case of a right-angled spherical triangle, consider which of the three quantities named (the two things given and the thing required) must be made the middle part, so that the other two may be equidistant from it, that is, both adjacent or both opposite parts; then one of the theorems contained in the prop. will give the value of the part required. Thus, if AC and CB be given to find A; then, if BC be made the middle part, AC and A will be the opposite parts; therefore $R \times s. BC = \cos. (90^\circ - A) \times \cos. (90^\circ - AC) = s. A \times s. AC$, therefore $s. A = \frac{s. BC}{s. AC}$, $R = 1$. Again, if AC and A be given to find AB, it is obvious that A is in the middle between the adjacent parts AB and $90^\circ - AC$; therefore $R \times \cos. A = \tan. AB \times \cot. AC$, therefore $\tan. AB = \frac{\cos. A}{\cot. AC} = \cos. A \times \tan. AC$ (42 Pl. Tr.).

In the same manner may all the other cases be resolved.

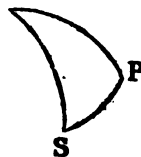
But without the consideration of the rule the equation may be immediately taken from the table, by observing which of the five cases contains the two given quantities and the quantity required. For instance, if AB and BC be given to find C; these three quantities are found in case 5, and $R \times \cos. AC = \cos. AB \times \cos. BC$, therefore $\log. \cos. AC = \log. \cos. AB + \log. \cos. BC - 10$. The value of the unknown quantity is thus immediately obtained from the equation in the table.

If you want to know the relative values of the quantities, you must convert the equation containing them into a proportion.

Quadrantal Spherical Triangles.

103. If the triangle DFE (fig. page 80) be the supplement of ABC, which is supposed to have the angle B right, then one side DF will be a quadrant (34); therefore DFE is called a *quadrantal triangle*.

As the sine, cosine, and tangent of an arc are the same in magnitude as the sine, cosine, and tangent of its supplement, the equations for the triangle ABC will apply to its supplemental triangle DEF. Therefore a quadrantal triangle may be resolved by the preceding rules, if we consider as circular parts, the two angles adjacent to the side of 90° , the complement of the other angle, and the complements of the other two sides. Hence, in the triangle ZPS, if the side ZS = 90° , the circular parts will succeed one another in this order; the angle Z, the comp. ZP, the comp. P, the comp. SP, the angle S. Consequently all the equations for the solution of a quadrantal triangle are contained in the following table.



Mid. Part	Equation
1. Z	$R \times s. Z = \begin{cases} \tan. S \times \cot. ZP \\ s. P \times s. SP \end{cases}$
2. S	$R \times s. S = \begin{cases} \tan. Z \times \cot. SP \\ s. P \times s. ZP \end{cases}$
3. comp. ZP	$R \times \cos. ZP = \begin{cases} \tan. Z \times \cot. P \\ \cos. S \times s. SP \end{cases}$
4. comp. SP	$R \times \cos. SP = \begin{cases} \tan. S \times \cot. P \\ \cos. Z \times s. ZP \end{cases}$
5. comp. P	$R \times \cos. P = \begin{cases} \cot. SP \times \cot. ZP \\ \cos. S \times \cos. P \end{cases}$

104. When any two parts are given to find a third, the case may be resolved by the rules in art. 100, or by seeking that equation in the table, which contains the three parts. If SP and ZP be given to find the angle P, these three parts are found in case 5, and the equation is

$R \times \cos. P = \cot. SP \times \cot. ZP$; therefore
 $\log. \cos. P = \log. \cot. SP + \log. \cot. ZP - 10.$

105. *Def.* The angle opposite to the quadrantal side is called the hypotenusal angle, and the other parts are called simply the sides and angles.

Affections of a Quadrantal Triangle.

106. The sides are of the same kind as their opposite angles, and conversely.

107. The hypotenusal angle is greater or less than 90° , according as a side and its adjacent angle, or the two sides, or the other two angles, are like or unlike.

108. An angle at the quadrant (or arc of 90°) is obtuse or acute, according as its adjacent side and the hypotenusal angle, or the other angle and the hypotenusal angle, are like or unlike.

109. A side is greater or less than 90° , according as its adjacent angle and the hypotenusal angle, or the other side and the hypotenusal angle, are like or unlike.

Other Properties of Quadrantal Triangles.

110. If the hypotenusal angle be 90° , one of the other angles and its opposite side will be each 90° ; and the other side and angle will be measured by the same number of degrees.

111. If an angle, or a side, be 90° , the opposite side, or angle, and the hypotenuse will be each 90° ; and the other angle and side will be measured by the same number of degrees.

112. If an angle at the quadrant be less than the hypotenusal angle, their sum will be less than 180° ; and if greater, their sum will be greater than 180° .

113. If a side be less than its opposite angle, their sum will be less than 180° ; and if greater, their sum will be greater than 180° .

114. The difference of the two sides is less than 90° , and their sum is greater than 90° , and less than 270° .

115. Each of the three angles is equal to or less than 90° ; or two angles are greater than 90° , and the third angle is less than 90° .

116. The proofs of these properties follow immediately from the properties of a right-angled spherical triangle (p. 80). If ABC be a spherical triangle, right-angled at B (fig. p. 80), then its supplemental triangle DEF will have one side DF,

opposite to B, equal to 90° : and those parts which are of the same or different affections in the triangle ABC, will have their corresponding parts of the same or different affections in the triangle DEF. By substituting 180° — one side or angle in DEF for the supplemental angle or side in ABC, we shall obtain the limits of the angles and sides in DEF.

Because $R \times \cos. AC = \cos. AB \times \cos. BC$ (100), if AC be less than 90° , its cosine will be positive; therefore $R \times \cos. AC$, or $\cos. AB \times \cos. BC$, is positive. Hence AB, BC must be both less or both greater than 90° ; for in the first case their cosines will be positive, and in the second negative (35 Pl. Tr.); therefore in both cases their product will be positive, as it ought. If AC be greater than 90° , its cosine is negative; therefore $\cos. AB \times \cos. BC$ is negative. Hence one of the sides AB, BC must be less than 90° , and the other greater; for then the cosine of one will be negative, and the cosine of the other positive; therefore their product will be negative, as it ought.

Again, $R \times s. BC = \cot. C \times \tan. AB$ (100). Now s. BC is always positive, therefore $\cot. C \times \tan. AB$ is always positive; consequently AB and C must both be less or both greater than 90° ; for in the first case the signs of $\cot. C$ and $\tan. AB$ will be positive, and in the second negative; therefore in both cases their product will be positive, as it ought. Hence the angles are of the same affections as their opposite sides.

117. Both in right-angled and in quadrantal triangles, if the value of the quantity required be expressed by its sine, it is impossible to determine whether that quantity ought to be less or greater than a quadrant; for the sine of an arc is the same as the sine of its supplement (34 Pl. Tr.). Hence the case is said to be ambiguous. In a right-angled spherical triangle the case is ambiguous when a side and its opposite angle are given to find any of the other parts; and in a quadrantal triangle the case is ambiguous when a side and its opposite angle are given to find any of the other parts, the given side not being the quadrantal side.

In all other cases the ambiguity is taken away by considering whether the sign of the quantity required is positive or negative (110 Pl. Tr.). In the equation $R \times \cos. C = \cot. AC \times \tan. BC$, if the angle C and the side AC be given, and BC be required, then $\tan. BC = \frac{R \times \cos. C}{\cot. AC}$. Now if C and AC be of the same affection, $\tan. BC$ is +, and therefore BC is less than 90° ; but if C and AC be of different affections, $\tan. BC$ is —, and therefore BC is greater than 90° .

In like manner, in all other cases where the quantity required is not expressed by its sine, the cosine, tangent, and cotangent indicate an arc less or greater than 90° , according as the sign of those quantities is positive or negative.

The ambiguity is often taken away by this property of a right-angled spherical triangle, that the sides are of the same affections as their opposite angles (35).

Solution of the Cases of Right-angled Spherical Triangles.

118. In a right-angled spherical triangle, of the three sides and the three angles any two being given, beside the right angle, which is a constant quantity, the other three may be found.

The different cases, or the varieties, which may happen in the solution of right-angled spherical triangles are sixteen. But these varieties may be reduced to six, if they be restricted to such as depend upon the same principles for their solution.

Given, to find the other parts,

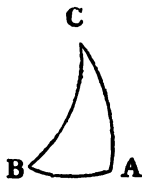
1. The hypotenuse and an angle,
2. A side and its adjacent angle,
3. A side and its opposite angle,
4. The hypotenuse and a side,
5. The two sides,
6. The two oblique angles.

119. These six cases may be resolved by the different propositions for the resolution of right-angled spherical triangles, or by the three following general theorems, which can be remembered more easily than if they were expressed separately.

1. $R \times s.$ either side $= \begin{cases} s. \text{ its op. angle} \times s. \text{ hyp.} \\ \cot. \text{ its adj. angle} \times \tan. \text{ other side.} \end{cases}$
2. $R \times \cos.$ either angle $= \begin{cases} \cos. \text{ its op. side} \times s. \text{ other angle.} \\ \tan. \text{ its adj. side} \times \cot. \text{ hyp.} \end{cases}$
3. $R \times \cos. \text{ hyp.} = \begin{cases} \cos. \text{ one side} \times \cos. \text{ other side.} \\ \cot. \text{ one angle} \times \cot. \text{ other angle.} \end{cases}$

To apply these equations to the solution of every case of right-angled triangles, they must be converted into analogies.

120. In the following table the first column contains the parts given, the second, the parts required, and the third, the proportions by which they are found.



GIVEN	REQ.	SOLUTION
I. BC B	AC	$R : \text{sine } BC :: \text{sine } B : \text{sine } AC$ (48). 1
	AB	$R : \cos. B :: \tan. BC : \tan. AB$ (53). 2
	C	$R : \cos. BC :: \tan. B : \cot. C$ (51). 3
II. AC C	AB	$R :: \text{sine } AC :: \tan. C : \tan. AB$ (49). 4
	BC	$\cos. C : R :: \tan. AC : \tan. BC$ (53). 5
	B	$R : \cos. AC :: \text{sine } C : \cos. B$ (57). 6
III. AC B	AB	$\tan. B :: \tan. AC :: R : \text{sine } AB$ (49). 7
	BC	$\text{sine } B : \text{sine } AC :: R : \text{sine } BC$ (48). 8
	C	$\cos. AC : \cos. B :: R : \text{sine } C$ (57). 9
IV. AC BC	AB	$\cos. AC : \cos. BC :: R : \cos. AB$ (56). 10
	B	$\text{sine } BC : \text{sine } AC :: R : \text{sine } B$ (48). 11
	C	$\tan. BC : \tan. AC : R : \cos. C$ (53). 12
V. AB AC	BC	$R : \cos. AB :: \cos. AC : \cos. BC$ (56). 13
	B	$\text{sine } AB : R :: \tan. AC : \tan. B$ (49). 14
	C	$\text{sine } AC : R :: \tan. AB : \tan. C$ (49). 14
VI. B C	AB	$\text{sine } B : \cos. C :: R : \cos. AB$ (57). 15
	AC	$\text{sine } C : \cos. B :: R : \cos. AC$ (57). 15
	BC	$\tan. B : \cot. C :: R : \cos. BC$ (51). 16

TABLE for determining the affections of the sides and angles found by the preceding analogies.

I. AC and B are of the same affection (35).	1
If $BC < 90^\circ$, AB and B are of the same affection; otherwise different (36).	2
If $BC < 90^\circ$, C and B are of the same affection; otherwise different (36).	3
II. AB and C are of the same affection (35).	4
If AC and C are of the same affection, $BC < 90^\circ$; otherwise, $BC > 90^\circ$ (36).	5
B and AC are of the same affection (35).	6
III. Ambiguous (117).	7
Ambiguous.	8
Ambiguous.	9
IV. When $BC < 90^\circ$, AB and AC are of the same affection; otherwise of different affections (36).	10
AC and B are of the same affection.	11
When $BC < 90^\circ$, AC and C are of the same affection; otherwise of different affections (36).	12
V. $BC < 90^\circ$, when AB and AC are of the same affection (36).	13
B and AC are of the same affection.	14
C and AB are of the same affection.	14
VI. AB and C are of the same affection.	15
AC and B are of the same affection.	16
When B and C are of the same affection, $BC < 90^\circ$; otherwise, $BC > 90^\circ$ (36).	16

121. *Remark.* In plane trigonometry, if the three angles of a triangle be given, the ratio of the sides may be found, but not their absolute values. In spherical trigonometry, if the three angles of a triangle be given, the sides can be determined from them.

Another remarkable difference subsists between plane and spherical triangles. In the former, when two angles are known, the third angle also is given; but in the latter, all the angles are independent of one another, and therefore must be found separately.

Solution of the Cases of Oblique-angled Spherical Triangles.

122. In any oblique-angled spherical triangle, of the three sides and the three angles any three being given, the other three may be found.

The different cases or varieties which may happen in the solution of oblique-angled spherical triangles are twelve. But these varieties may be reduced to six, if they be restricted to such as depend upon the same principles for their solution.

Given, to find the other parts,

1. Two sides and an angle opposite to one of them,
2. Two angles and a side opposite to one of them,
3. Two sides and the included angle,
4. Two angles and the side between them,
5. The three sides,
6. The three angles.

123. If a perpendicular arc be drawn from one of the angles upon the opposite side, all the cases of oblique-angled triangles, except the two where the three sides or the three angles are given, may be resolved by means of the preceding analogies for the resolution of right-angled triangles.

But all the cases of oblique-angled spherical triangles may be resolved without a perpendicular, by means of the following theorems, which are better adapted to practice, and less difficult to remember, than the various particulars which must be attended to in the first method, with respect to the falling of the perpendicular, and the species of the different parts of the triangle.

1. The sine of any side
 : sine of its opposite angle
 :: sine of any other side
 : sine of its opposite angle.
2. The sine $\frac{1}{2}$ sum of any two sides
 : sine $\frac{1}{2}$ their difference
 :: cot. $\frac{1}{2}$ included angle
 : tan. $\frac{1}{2}$ difference of the other two angles.
3. The cos. $\frac{1}{2}$ sum of any two sides
 : cos. $\frac{1}{2}$ their difference
 :: cot. $\frac{1}{2}$ included angle
 : tan. $\frac{1}{2}$ sum of the other two angles.
4. The sine $\frac{1}{2}$ sum of any two angles
 : sine $\frac{1}{2}$ their difference
 :: tan. $\frac{1}{2}$ included side
 : tan. $\frac{1}{2}$ difference of the other two sides.
5. The cos. $\frac{1}{2}$ sum of any two angles
 : cos. $\frac{1}{2}$ their difference
 :: tan. $\frac{1}{2}$ included side
 : tan. $\frac{1}{2}$ sum of the other two sides.
6. The rectangle under the sines of any two sides
 : R^2
 :: sine $\frac{1}{2}$ sum of the three sides $\times$ sine dif.
 between the half sum and the third side
 : $\cos.^2 \frac{1}{2}$ included angle.
7. The rectangle under the sines of any two angles
 : R^2
 :: cos. $\frac{1}{2}$ sum of the three angles $\times$ cos. dif. between
 the half sum and the third angle
 : $\sin^2 \frac{1}{2}$ included angle.

All these theorems are adapted to logarithmic computation, and are applicable to any spherical triangle.

Theorems 2, 3, 4, 5 may be varied by inversion, or alternation. Thus, theorem 2 becomes, by inversion,

- sine $\frac{1}{2}$ difference of two sides
 : sine $\frac{1}{2}$ their sum
 :: tan. $\frac{1}{2}$ difference of two angles
 : cot. $\frac{1}{2}$ angle included between the two sides.

This analogy finds the third angle, when the two sides including it, and the other two angles are known.

124. In the resolution of spherical triangles there is a source of error which, in some instances, ought not to be neglected. The error arises from the magnitude of an arc, and the trigonometrical term by which its value is expressed. Thus, if a great arc be expressed by a sine, or a small arc by a cosine, there may be an error in its value found by the solution.

An arc between 45° and 135° is commonly called a great arc; and an arc under 45° , or above 135° , is usually called a small arc. The former is sometimes called a mean arc, and the latter an extreme arc. But the error in a solution is of no consequence, unless an arc be very near 90° , and its value be expressed by a sine; or very near 0° , and its value be expressed by a cosine.

125. These vague or inaccurate solutions arise from the unequal manner in which the sine of an arc increases from 0° to 90° , or the cosine decreases; for the sine of a small arc increases by great increments, and the sine of a great arc by small increments; but the cosine of a small arc decreases by small increments, and the cosine of a great arc by great increments. A very small arc and its sine are very nearly equal; therefore the sine will increase nearly as fast as the arc. The sine of a great arc is nearly perpendicular to the arc; therefore it increases very little by a small change of its situation. Hence the sines near 90° increase by small increments, while those near 0° increase by great increments. The contrary is the case with respect to the cosines.

The unequal variation of the sines and cosines of arcs is observable in the log. tables. The log. difference between the sines of the seconds is great near 0° , but very small near 90° , where there is no difference of the sines for nearly 100 seconds.

126. Hence it is manifest that, if a great arc be expressed by a sine, or a small arc by a cosine, its value cannot be determined with certainty. To avoid such a doubtful solution the following rule must be observed. *If an arc be great its value must not be expressed by a sine; and if an arc be very small, its value must not be expressed by a cosine.*

The log. differences of the tangents and cotangents are considerable in all parts of an arc; therefore if the value of any arc be expressed by a tangent or a cotangent, no error will arise in the solution of a problem.

Hence, if any vague solution be likely to occur from a sine or a cosine, the denomination must be changed to a tangent or

a cotangent; that is, the value of a sine or a cosine must be expressed by a tangent or a cotangent, and be substituted for the former in the result.

127. *Numerical Solution of the Cases of Right-angled Spherical Triangles.*

Case 1. Given the hypotenuse BC $64^{\circ} 40'$, and the angle C 48° , to find the rest.

R or s.	90°	-	-	-	10.
S. BC	$64^{\circ} 40'$	-	-	-	9.9560886
S. C	48°	-	-	-	9.8710735

S. AB	$42^{\circ} 12'$	-	-	-	9.8271621
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AB is less than 90° , because its op. angle C is acute (35).

R	-	-	-	-	10.
Cos. C	48°	-	-	-	9.8255109
Tan. BC	$64^{\circ} 40'$	-	-	-	10.3246995

Tan. AC	$54^{\circ} 43'$	-	-	-	10.1502104
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AC is less than 90° , because BC and C are like (37).

R	-	-	-	-	10.
Cos. BC	$64^{\circ} 40'$	-	-	-	9.6313258
Tan. C	48°	-	-	-	10.0455626

Cot. B	$64^{\circ} 35'$	-	-	-	9.6768884
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B is acute, because BC and C are like (38).

Case 2. Given the side AC $54^{\circ} 43'$, and its adjacent angle C 48° , to find the rest.

R	-	-	-	-	10.
S. AC	$54^{\circ} 43'$	-	-	-	9.9118528
Tan. C	48°	-	-	-	10.0455626

Tan. AB	$42^{\circ} 12'$	-	-	-	9.9574154
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AB is less than 90° , because its op. angle C is acute (35).

Cos. C	48°	-	-	-	9·8255109
R	-	-	-	-	10·
Tan. AC	54° 43'	-	-	-	10·1502104

Tan. BC	64° 40'	-	-	-	10·3246995
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BC is less than 90°, because AC and C are like (36).

R	-	-	-	-	10·
Cos. AC	54° 43'	-	-	-	9·7614638
S. C	48°	-	-	-	9·8710735

Cos. B	64° 35'	-	-	-	9·6325373
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B is acute, because its op. side AC is less than 90°.

Case 3. Given the side AB 42° 12', and its opposite angle C 48°, to find the rest.

Tan. C	48°	-	-	-	10·0455626
Tan. AB	42° 12'	-	-	-	9·9574850
R	-	-	-	-	10·

S. AC	54° 43' or 125° 17'	-	-	-	9·9119224
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S. C	48°	-	-	-	9·8710735
S. AB	42° 12'	-	-	-	9·8271887
R	-	-	-	-	10·

S. BC	64° 40' or 115° 20'	-	-	-	9·9561152
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Cos. AB	42° 12'	-	-	-	9·8697037
Cos. C	48°	-	-	-	9·8255109
R	-	-	-	-	10·

S. B	64° 35' or 115° 25'	-	-	-	9·9558072
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Note. In the remaining cases the analogies and the answers are given; and the logarithmic operations must be performed by the learner.

Case 4. Given the hypotenuse BC 64° 40', and a side AB 42° 12', to find the rest.

Cos. AB $42^{\circ} 12'$: cos. BC $64^{\circ} 40'$:: R : cos. AC $54^{\circ} 43'$.
AC is less than 90° , because AB and BC are like (37).

S. BC $64^{\circ} 40'$: s. AB $42^{\circ} 12'$:: R : s. C 48° .
C is acute, because its op. side is less than 90° .

Tan. BC $64^{\circ} 40'$: tan. AB $42^{\circ} 12'$:: R : cos. B $64^{\circ} 35'$.
B is acute, because BC and B are like (38).

Case 5. Given the side AC $54^{\circ} 43'$, and AB $42^{\circ} 12'$, to find the rest.

R : cos. AC $54^{\circ} 43'$:: cos. AB $42^{\circ} 12'$: cos. BC $64^{\circ} 40'$.
BC is less than 90° , because AB and AC are like (36).

S. AB $42^{\circ} 12'$: R :: tan. AC $54^{\circ} 43'$: tan. B $64^{\circ} 35'$.
B is acute, because it is like its op. side AC.

S. AC $54^{\circ} 43'$: R :: tan. AB $42^{\circ} 12'$: tan. C 48° .
C is acute, being like its op. side AB.

Case 6. Given the two oblique angles B $64^{\circ} 35'$, and C 48° , to find the sides.

S. B $64^{\circ} 35'$: cos. C 48° :: R : cos. AB $42^{\circ} 12'$.
AB is less than 90° , being like its op. angle C.

S. C 48° : cos. B $64^{\circ} 35'$:: R : cos. AC $54^{\circ} 43'$.
AC is less than 90° , being like its op. angle B.

Tan. B $64^{\circ} 35'$: cot. C 48° :: R : cos. BC $64^{\circ} 40'$.
BC is less than 90° , because B and C are like (36).

Miscellaneous Examples.

1. Given the hyp. BC $78^{\circ} 20'$, and the side AB $36^{\circ} 31'$, to find the rest.

Answer. AC = $75^{\circ} 25'$, B = $81^{\circ} 12'$, C = $37^{\circ} 25'$.

2. Given the side AC $27^{\circ} 54'$, and AB $11^{\circ} 30'$, to find the rest.

Answer. BC = 30° , B = $69^{\circ} 22'$, C = $23^{\circ} 30'$.

3. Given the two oblique angles, B $81^{\circ} 12'$, and C $37^{\circ} 25'$, to find the rest.

Answer. AB = $36^{\circ} 31'$, AC = $75^{\circ} 25'$, BC = $78^{\circ} 20'$.

4. Given the side AB $11^{\circ} 30'$, and its opposite angle C $23^{\circ} 30'$, to find the rest.

Answer. $AC = 27^\circ 54'$ or $152^\circ 6'$, $B = 69^\circ 22'$ or $110^\circ 38'$, $BC = 30^\circ$ or 150° .

5. Given the side $AC 75^\circ 25'$, and its adjacent angle $C 37^\circ 25'$, to find the rest.

Answer. $AB = 36^\circ 31'$, $B = 81^\circ 12'$, $BC = 78^\circ 20'$.

6. Given the hyp. $BC 30^\circ$, and the angle $C 23^\circ 30'$, to find the rest.

Answer. $AB = 11^\circ 30'$, $AC = 27^\circ 54'$, $B = 69^\circ 22'$.

7. In the right-angled spherical triangle ABC one of the oblique angles B is 60° , and the other C is 45° . Required the side AC opposite to the greater angle B . *Answer.* $AC = 45^\circ$.

8. In a right-angled isosceles spherical triangle each of the equal sides is 30° . Required the hypotenuse.

Answer. $41^\circ 24' 35''$.

128. The following problems are solved by Napier's Rules, which may be applied to the preceding examples. *Plate, Fig.*

1. In the right-angled spherical triangle ABC , given one side $AB 49^\circ 17'$, and its adjacent angle $A 23^\circ 28'$, to find the other side BC , and the hyp. AC .

Here AB is middle part, and BC and comp. A are adjacent parts; therefore $R \times s. AB = \tan. BC$ and $\cot. A$ (100), therefore $\log. \tan. BC = \log. R + \log. s. AB - \log. \cot. A$.

$$\begin{array}{rcll} 10 + \log. s. & 49^\circ 17' & - & - & 19.8796375 \\ \log. \cot. & 23^\circ 28' & - & - & 10.3623894 \end{array}$$

$$\log. \tan. BC = 18^\circ 12' 48'' \quad - \quad 9.5172481$$

, Again, comp. A middle part, and AB and comp. AC adjacent parts; therefore $R \times \cos. A = \tan. AB$ and $\cot. AC$, therefore $\log. \cot. AC = \log. R + \log. \cos. A - \log. \tan. AB$.

$$\begin{array}{rcll} 10 + \log. \cos. & 23^\circ 28' & - & - & 19.9625076 \\ \log. \tan. & 49^\circ 17' & - & - & 10.0651775 \end{array}$$

$$\log. \cot. AC = 51^\circ 42' 37'' \quad - \quad 9.8973301$$

2. Given $PS = 68^\circ 43'$, and $ZS = 72^\circ 16'$, to find PZ and the angle Z . *Fig. page 106.*

Here comp. ZS middle part, and ZP , PS opposite parts; therefore $R \times \cos. ZS = \cos. ZP \times \cos. PS$; therefore $\log. \cos. PZ = \log. R + \log. \cos. ZS - \log. \cos. PS$.

$$\begin{array}{rcll} 10 + \log. \cos. & 72^\circ 16' & - & - & 19.4837117 \\ \log. \cos. & 68^\circ 43' & - & - & 9.5598829 \end{array}$$

$$\log. \cos. PZ = 57^\circ 2' 54'' \quad - \quad 9.9238288$$

Again, PS middle part, and comp. ZS and comp. Z op. parts; therefore $R \times s. PS = s. ZS \times s. Z$; therefore $\log. s. Z = \log. R + \log. s. PS - \log. s. ZS$.

$$\begin{array}{rcl} 10 + \log. s. 68^\circ 43' & - & - & 19.9693212 \\ \log. s. 72^\circ 16' & - & - & 9.9788579 \\ \hline \end{array}$$

$$\log. s. Z = 78^\circ 2' 12'' \quad - \quad 9.9904633$$

3. Given PS $66^\circ 32'$, and PZ $37^\circ 48'$, to find ZS.

Here comp. ZS middle part, and PZ and PS op. parts; therefore $R \times \cos. ZS = \cos. PZ \times \cos. PS$; therefore $\log. \cos. ZS = \log. \cos. PZ + \log. \cos. PS - 10$.

$$\begin{array}{rcl} \log. \cos. & 37^\circ 48' & - & - & 9.8977123 \\ \log. \cos. & 66^\circ 32' & - & - & 9.6001181 \\ \hline \end{array}$$

$$\log. \cos. ZS = 71^\circ 39' 37'' \quad - \quad 9.4978304$$

4. In the quadrantal triangle ZPS, where $ZS = 90^\circ$, given ZP $37^\circ 48'$, and SP $66^\circ 32'$, to find the angles P and Z.

Here comp. P middle part, and comp. ZP and comp. SP adj. parts; therefore $R \times \cos. P = \cot. ZP \times \cot. SP$; therefore $\log. \cos. P = \log. \cot. ZP + \log. \cot. SP - 10$.

$$\begin{array}{rcl} \log. \cot. & 37^\circ 48' & - & - & 10.1103177 \\ \log. \cot. & 66^\circ 32' & - & - & 9.6376106 \\ \hline \end{array}$$

$$\log. \cos. P = 55^\circ 58' 2'' \quad - \quad 9.7479283$$

The angle in the tables answering to $\log. \cos. P$ is $55^\circ 58' 2''$: but since PZ and PS are like, the angle P must be greater than 90° (107); therefore we must take the supplement of P, or $124^\circ 1' 58''$, for the angle sought.

Again, comp. PS middle part, and Z and comp. PZ op. parts; therefore $R \times \cos. PS = s. PZ \times \cos. Z$; therefore $\log. \cos. Z = 10 + \log. \cos. PS - \log. s. PZ$.

$$\begin{array}{rcl} 10 + \log. \cos. & 66^\circ 32' & - & - & 19.6001181 \\ \log. s. & 37^\circ 48' & - & - & 9.7873946 \\ \hline \end{array}$$

$$\log. \cos. Z = 49^\circ 28' 47'' \quad - \quad 9.8127235$$

From these examples the reader will perceive that Napier's Rules apply with facility to all the cases of right-angled spherical triangles. As they are concise and easy to remember, they are better adapted to the solution of numerical examples than the analogies in the preceding table.

129. *Numerical Solution of the Cases of Oblique-angled Spherical Triangles.* Plate, Fig. 6.

Case 1. In the oblique-angled spherical triangle ABC, given the side AB $80^{\circ} 19'$, the side AC $63^{\circ} 50'$, and the opposite angle B $51^{\circ} 30'$, to find the rest.

S. AC $63^{\circ} 50'$	-	-	-	9.9530418
S. B $51^{\circ} 30'$	-	-	-	9.8935444
S. AB $80^{\circ} 19'$	-	-	-	9.9937679

S. C $59^{\circ} 16'$ or $120^{\circ} 44'$	-	-	-	9.9342705
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S. $\frac{1}{2}$ (AB — AC) $8^{\circ} 14'\frac{1}{2}$	-	-	-	9.1564435
---	---	---	---	-----------

S. $\frac{1}{2}$ (AB + AC) $72^{\circ} 4'\frac{1}{2}$	-	-	-	9.9783906
---	---	---	---	-----------

Tan. $\frac{1}{2}$ (C — B) $3^{\circ} 53'$	-	-	-	8.8317478
--	---	---	---	-----------

Cot. $\frac{1}{2}$ A $65^{\circ} 46'$	-	-	-	9.6536949
---------------------------------------	---	---	---	-----------

Therefore the angle A = $131^{\circ} 32'$.

If C had been taken $120^{\circ} 44'$, A would have been $24^{\circ} 36'$.

S. $\frac{1}{2}$ (C — B) $3^{\circ} 53'$	-	-	-	8.8307495
--	---	---	---	-----------

S. $\frac{1}{2}$ (C + B) $55^{\circ} 23'$	-	-	-	9.9153846
---	---	---	---	-----------

Tan. $\frac{1}{2}$ (AB — AC) $8^{\circ} 14'\frac{1}{2}$	-	-	-	9.1609021
---	---	---	---	-----------

Tan. $\frac{1}{2}$ BC $60^{\circ} 23'$	-	-	-	10.2455372
--	---	---	---	------------

Therefore BC = $120^{\circ} 46'$.

If C had been taken $120^{\circ} 44'$, BC would have been $28^{\circ} 34'$.

This example affords an opportunity of remarking, that the angle C opposite to the other given side, is always either an acute angle or its supplement. But it appears from the calculation that this may not be the case with respect to the remaining side or the remaining angle; for the two values of A are $131^{\circ} 32'$ and $24^{\circ} 36'$, and the two values of BC are $120^{\circ} 46'$ and $28^{\circ} 34'$, which two values, in both instances, are not supplements of each other.

Case 2. Given the angle A $59^{\circ} 16'$, the angle C $51^{\circ} 30'$, and its opposite side AB $63^{\circ} 50'$, to find the rest.

S. C $51^{\circ} 30'$	-	-	-	9.8935444
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S. AB $63^{\circ} 50'$	-	-	-	9.9530418
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S. A $59^{\circ} 16'$	-	-	-	9.9342737
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s. BC $80^{\circ} 19'$ or $99^{\circ} 41'$	-	-	-	9.9937711
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As the angle A is greater than C, the side BC will also be greater than AB (26). Now both the values of BC are greater than the value of AB, and therefore the answer is ambiguous.

$$\begin{array}{llll} \text{S. } \frac{1}{2} (A - C) & 3^\circ 53' & - & 8.8307495 \\ \text{S. } \frac{1}{2} (A + C) & 55^\circ 23' & - & 9.9153846 \\ \text{T. } \frac{1}{2} (BC - AB) & 8^\circ 14' \frac{1}{2} & - & 9.1609021 \end{array}$$

$$\text{T. } \frac{1}{2} AC \quad 60^\circ 23' \frac{1}{2} \quad - \quad 10.2455372$$

Therefore $AC = 120^\circ 47'$.

If BC had been taken $99^\circ 41'$, AC would have been $151^\circ 27'$.

$$\begin{array}{llll} \text{S. } \frac{1}{2} (BC - AB) & 8^\circ 14' \frac{1}{2} & - & 9.1563935 \\ \text{S. } \frac{1}{2} (BC + AB) & 72^\circ 4' \frac{1}{2} & - & 9.9788906 \\ \text{T. } \frac{1}{2} (A - C) & 3^\circ 53' & - & 8.8317478 \end{array}$$

$$\text{Cot. } \frac{1}{2} B \quad 65^\circ 45' \quad - \quad 9.6537449$$

Therefore the angle $B = 131^\circ 30'$.

If BC had been taken $99^\circ 41'$ the angle B would have been $152^\circ 22'$.

From this example it appears that the side BC first found, must be either an arc less than 90° ($80^\circ 19'$), or its sup. ($99^\circ 41'$); and that the two values of the side AC ($120^\circ 47'$ and $151^\circ 27'$) are obtuse, and also the two values of the angle B ($131^\circ 30'$ and $152^\circ 22'$). But these two values, in both instances, are not supplements of each other.

Case 3. Given two sides AB $80^\circ 19'$ and BC $120^\circ 47'$, and the included angle B $51^\circ 30'$, to find the rest.

$$\begin{array}{llll} \text{An. Cot. } \frac{1}{2} (AB + BC) & 100^\circ 33' & - & 9.2626729 \\ \text{✓ Cot. } \frac{1}{2} (BC - AB) & 20^\circ 14' & - & 9.9723380 \\ \text{Cot. } \frac{1}{2} B & 25^\circ 45' & & 10.3166443 \end{array}$$

$$\text{T. } \frac{1}{2} (A + C) \quad 84^\circ 37' \text{ or } 95^\circ 23' \quad 11.0263094$$

In this example the latter number $95^\circ 23'$ must be taken to make the half sum of the angles A and C agree with the half sum of their opposite sides.

$$\begin{array}{llll} \text{S. } \frac{1}{2} (BC + AB) & 100^\circ 33' & - & 9.9925957 \\ \text{S. } \frac{1}{2} (BC - AB) & 20^\circ 14' & - & 9.5388804 \\ \text{Cot. } \frac{1}{2} B & 25^\circ 45' & - & 10.3166443 \end{array}$$

$$\text{T. } \frac{1}{2} (A - C) \quad 36^\circ 7' \quad - \quad 9.8629290$$

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Hence $A = 36^\circ 7' + 95^\circ 23' = 131^\circ 30'$, and $C = 95^\circ 23' - 36^\circ 7' = 59^\circ 16'$.

S. $\frac{1}{2} (A - C)$	$36^\circ 7'$	-	9.7704332
S. $\frac{1}{2} (A + C)$	$95^\circ 23'$	-	9.9980802
T. $\frac{1}{2} (BC - AB)$	$20^\circ 14'$	-	9.5665424

T. $\frac{1}{2} AC$	$31^\circ 55'$	-	9.7941894
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Therefore $AC = 63^\circ 50'$.

Case 4. Given two angles $A 131^\circ 30'$ and $B 51^\circ 30'$, and the included side $AB 80^\circ 19'$, to find the rest.

Cos. $\frac{1}{2} (A + B)$	$91^\circ 30'$	-	8.4179190
Cos. $\frac{1}{2} (A - B)$	40°	-	9.8842540
T. $\frac{1}{2} AB$	$40^\circ 9' \frac{1}{2}$	-	9.9262496

T. $\frac{1}{2} (AC + CB)$	$87^\circ 41'$ or $92^\circ 19'$	-	11.3925846
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The greater number $92^\circ 19'$ must be taken, to make the half sum of the sides agree with the half sum of their opposite angles.

S. $\frac{1}{2} (A + B)$	$91^\circ 30'$	-	9.9998512
S. $\frac{1}{2} (A - B)$	40°	-	9.8080675
T. $\frac{1}{2} AB$	$40^\circ 9' \frac{1}{2}$	-	9.9262496

T. $\frac{1}{2} (BC - AC)$	$28^\circ 28'$	-	9.7344659
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Therefore $BC = 92^\circ 19' + 28^\circ 28' = 120^\circ 47'$, and $AC = 92^\circ 19' - 28^\circ 28' = 63^\circ 51'$.

S. $\frac{1}{2} (BC - AC)$	$28^\circ 28'$	-	9.6781972
S. $\frac{1}{2} (BC + AC)$	$92^\circ 19'$	-	9.9996398
T. $\frac{1}{2} (A - B)$	40°	-	9.9238135

Cot. $\frac{1}{2} C$	$29^\circ 38'$	-	10.2452561
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Therefore the angle $C = 59^\circ 16'$.

Case 5. Given the three sides, $AB 80^\circ 19'$, $BC 120^\circ 47'$, $AC 63^\circ 50'$, to find the angles.

$$132^\circ 28' = \frac{1}{2} \text{ sum of the sides}$$

$$68^\circ 38' = \frac{1}{2} \text{ sum} - AC$$

S. AB	80° 19'	-	-	-	9.9937679
S. BC	120° 47'	-	-	-	9.9340482

Sum - 19.9278161

			Comp.	0.0721839
S. $\frac{1}{2}$ sum	132° 28'	-		9.8678623
S. ($\frac{1}{2}$ sum — AC)	68° 38'	-		9.9690746

2) 19.9091208

Cos. $\frac{1}{2}$ B 25° 45' - - - 9.9545604

Therefore the angle B = 51° 30'.

In the same manner the angle C is found 59° 16', and the angle A 131° 30'. Or, the angles may be found by the rule for sine $\frac{1}{2}$ B, page 102.

But when one angle is found by this rule, the other two angles may be more readily determined by the first case.

In the triangle ZPS (fig. page 106) given ZP = 37° 48', PS = 74° 30', and ZS = 108°, to find the angle P.

110° 9' = $\frac{1}{2}$ sum of the sides

2° 9' = $\frac{1}{2}$ sum — ZS

S. PS	74° 30'	-	-	-	9.9839105
S. PZ	37° 48'	-	-	-	9.7873946

Sum 19.7713051

			Comp.	0.2286949
S. $\frac{1}{2}$ sum	110° 9'	-		9.9725703
S. ($\frac{1}{2}$ sum — ZS)	2° 9'	-		8.5742139

2) 18.7754791

Cos. $\frac{1}{2}$ P 75° 51' 56" 9.3877395

Therefore the angle P = 151° 43' 52".

Case 6. Given the three angles A 131° 30', B 51° 30', C 59° 16', to find the sides.

121° 8' = $\frac{1}{2}$ sum of the angles

61° 52' = $\frac{1}{2}$ sum — C

S. A	131° 30'	-	-	-	9·8744561
S. B	51° 30'	-	-	-	9·8935444

Sum	19·7680005
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		Comp.	0·2319995
Cos. $\frac{1}{2}$ sum	121° 8'	-	9·7135169
Cos. ($\frac{1}{2}$ sum — C)	61° 52'	-	9·6735047

2)	19·6190211
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S. $\frac{1}{2}$ AB	40° 9' $\frac{1}{2}$	-	-	9·8095105
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Therefore AB = 80° 19'.

In the same manner AC is found 63° 50', and BC 120° 47'.
Or, AB may be found by the rule for cos. $\frac{1}{2}$ AB. See page 102.

But when one side is found by this rule, the other two sides may be more readily determined by the second case.

Miscellaneous Examples.

1. Given two sides and an opposite angle, namely, AC 58°, BC 110°, and the angle B 50°, to find the rest.

Answer. A 121° 54' 56'', C 62° 34' 6'', AB 79° 17' 14''.

2. Given two sides and the included angle, namely, AC 58°, BC 110°, and the angle C 62° 34' 6'', to find the rest.

Answer. A 121° 54' 56'', B 50°, AB 79° 17' 14''.

3. Given two angles and an opposite side, namely, A 121° 54' 56'', B 50°, AC 58°, to find the rest.

Answer. BC 110°, AB 79° 17' 14'', C 62° 34' 6''.

4. Given two angles and the included side, namely, A 121° 54' 56'', C 62° 34' 6'', AC 58°, to find the rest.

Answer. BC 110°, AB 79° 17' 14'', B 50°.

5. Given two sides 57° 30' and 82° 26', and the included angle 126° 37', to find the other parts.

Answer. The angles 61° 40' and 48° 30', and the third side 115° 20'.

6. Given two angles 61° 40' and 126° 37', and the included side 57° 30', to find the other parts.

Answer. The other sides 82° 26' and 115° 20', and the third angle 48° 30'.

7. Given the three sides $65^{\circ} 30'$, $96^{\circ} 47' 50''$, $56^{\circ} 40'$, to find the angles.

Answer. $54^{\circ} 40'$, $48^{\circ} 30'$, $117^{\circ} 5' 56''$.

8. Given the three sides $114^{\circ} 30'$, $83^{\circ} 13'$, $50^{\circ} 40'$, to find the angles.

Answer. $48^{\circ} 31'$, $62^{\circ} 56'$, $125^{\circ} 19'$.

9. Given the three angles $48^{\circ} 31'$, $62^{\circ} 52'$, $125^{\circ} 20'$ to find the sides.

Answer. $114^{\circ} 29'$, $83^{\circ} 9'$, $56^{\circ} 42'$.

10. Given the three angles $61^{\circ} 40'$, $126^{\circ} 37'$, $48^{\circ} 30'$, to find the sides.

Answer. $57^{\circ} 30'$, $82^{\circ} 26'$, $115^{\circ} 20'$.

11. Each side of an equilateral triangle is 60° . Required the angles.

Answer. Each angle is $70^{\circ} 31' 44''$.

12. In an isosceles triangle each of the equal sides is $95^{\circ} \frac{1}{2}$, and their included angle is 100° . Required the base.

Answer. $99^{\circ} 22' 24''$.

13. Given two sides of a triangle $96^{\circ} 50'$ and $83^{\circ} 10'$, and their included angle 120° . Required the third side and the other two angles.

Answer. Third side $120^{\circ} 28' 10''$, and the angles $86^{\circ} 4' 13''$, and $93^{\circ} 55' 47''$.

APPENDIX.

TO find the Seconds in an Angle corresponding to any given Logarithm; and, conversely, to find the Logarithm answering to any Angle containing Seconds. See articles 84 and 138.

The seconds of a degree may be found by proportion as follows. Let 9.6699740 be the log. sine of an angle A. The nearest log. sine to this in the common tables is 9.6699420, which answers to $27^{\circ} 53'$ to the nearest minute. Now the difference between these two logarithms is 320; and in the tables it appears that the difference of the logarithms corresponding to $1''$ is 39.8. Hence $39.8 : 320 :: 1' : 8' (130 \text{ Pl. Tr.})$. Consequently the angle $A = 27^{\circ} 53' 8''$. In this manner we can find the angle to seconds corresponding to any logarithm.

In general, let $a^{\circ} b'$ represent any arc or angle, whose log. l is next *less* than the given log. L, if it be the log. of a sine, tangent, or secant; or next *greater*, if it be the log. of a cosine, cotangent, or cosecant; and let d be the difference of the logarithms corresponding to $1''$; then $d : L - l :: 1'' : c''$. Hence $a^{\circ} b' c''$ is the arc or angle corresponding to the given log. L.

On the contrary, we can find the logarithm answering to any angle containing seconds, as follows. Let the arc $a^{\circ} b' c''$ be given to find its logarithm, then $1'' : c'' : d : D$. Therefore $l + D$ is the log. required, if it be the log. of a sine, tangent, or secant; and $l - D$ is the log., if it be the log. of a cosine, cotangent, or cosecant.*

In small arcs the logarithmic sines and tangents increase so irregularly that the differences for seconds are not in the same ratio as the differences for minutes.

* The two following rules were communicated by Joseph Clay, Esq.

Now as a small arc decreases, and approximates to a right line, the differences of the logarithms of numbers expressing the different lengths of the decreasing arc approximate to the differences of the logarithms of the sines and tangents (130 Pl. Tr.). Therefore in small arcs containing seconds, the following rules give results much nearer the truth than the preceding rules by proportion.

To find the Logarithmic Sine or Tangent of any small Arc containing Seconds.

Reduce the degrees and minutes of the given arc to seconds, and find the logarithm of that number of seconds.

Reduce the whole arc to seconds, and find the logarithm of that number of seconds.

Subtract the first logarithm from the second.

Find the logarithmic sine or tangent of the degrees and minutes in the arc proposed, to which add the difference of the logarithms found above. The sum is the logarithmic sine or tangent of the given arc.

Required the log. tangent of $4^{\circ} 25' 37''$.

$$\begin{array}{rcl} 4^{\circ} 25' & = 15900'', \text{ whose log. is } & 4.201397 \\ 4^{\circ} 25' 37'' & = 15937'', \text{ whose log. is } & 4.202407 \end{array}$$

	Difference	
Log. tan. $4^{\circ} 25'$	-	8.887833
Log. tan. $4^{\circ} 25' 37''$ required	-	8.888843

Given the Logarithmic Sine or Tangent of any small Arc, to find the Arc.

Find the logarithmic sine or tangent next less than the given logarithm, and subtract it from the given logarithm.

Reduce the degrees and minutes answering to the logarithm next less than the given logarithm to seconds, and find the logarithm of that number of seconds.

To this last logarithm add the difference of the logarithms found above, and the number answering to the sum of these logarithms is the number of seconds in the arc required.

Required the arc answering to the log. sine 8.682553.

Given log. sine 8.682553

Log. sine next less 8.681043, whose arc is $2^{\circ} 45' = 9900''$

	Difference	·1510
Log. 9900	- -	3.995633
Sum	- -	3.997145

The number answering to this log. is $9934.5'' = 2^{\circ} 45' 34.5''$
the arc required.

Of the Tangent and Secant of an Arc of 90° .

Difficulties have arisen from the supposition that an arc of 90° has a tangent and a secant, both infinite. The following error arises from this supposition. In a right-angled spherical triangle, radius : cos. angle at the base :: tan. hyp. : tan. base (53). Now when the base is 90° , the hypotenuse is also 90° (40). Therefore if these two equal arcs have any tangents, they must be equal; therefore radius = cos. angle at the base, whatever that angle may be. This false conclusion arises from the supposition that an arc of 90° has a tangent.

As the arc increases to 90° , the tangent and secant increase without limit, cease to exist when the arc is 90° , and then begin again at a quantity indefinitely great.

Hence it follows that, when a tangent or a secant enters into a calculation, and the arc becomes 90° , we can draw no infallible conclusion from the calculation.

THE END.









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